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Torsional wave in an inhomogeneous prestressed elastic layer overlying an inhomogeneous elastic half-space under the effect of rigid boundary

Rajneesh Kakar*

163/1, C-B, Jalandhar-144022, India

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Abstract. An investigation has been carried out for the propagation of torsional surface waves in an inhomogeneous prestressed layer over an inhomogeneous half space when the upper boundary plane is assumed to be rigid. The inhomogeneity in density, initial stress (tensile and compressional) and rigidity are taken as an arbitrary function of depth, where as for the elastic half space, the inhomogeneity in density and rigidity is hyperbolic function of depth. In the absence of heterogeneities of medium, the results obtained are in agreement with the same results obtained by other relevant researchers. Numerically, it is observed that the velocity of torsional wave changes remarkably with the presence of inhomogeneity parameter of the layer. Curves are compared with the corresponding curve of standard classical elastic case. The results may be useful to understand the nature of seismic wave propagation in geophysical applications.

Keywords: torsional waves; rigid boundary; density; rigidity; in-homogeneity; initial stress

1. Introduction

The term “Prestress” is meant by stresses developed in a medium before it is being used for study. The earth is an initially stressed medium. Due to presence of external loading, slow process of creep and gravitational field, considerable amount of stresses (called pre-stresses or initial stresses) remain naturally present in the layers. These stresses may have significant influence on torsional waves produced by earthquake or explosions (Kakar and Kakar 2012). For seismologists, the propagation of torsional surface wave in elastic and viscoelastic layered media is useful to understand earthquake disaster prevention because during propagation, the said wave twists the medium. Due to inhomogeneity in the crust of the earth, the study of torsional wave becomes important to understand earthquake disaster. Also, torsional waves are affected by the material properties of the medium through which they travel. If the variation in elastic parameters of the medium is quite small, the main effect will be an attenuation of the seismic energy.

Much literature is available on the subject of surface waves such as Rayleigh and Love waves due to drastic capabilities during earthquake and practical applications in the field of geophysical prospecting; unfortunately little literature is available on torsional surface wave propagation in

*Corresponding author, Professor, E-mail: rajneesh.kakar@gmail.com

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inhomogeneous elastic and viscoelastic media. Dey *et al*. (1996) discussed the propagation of torsional surface waves with constant density and variable rigidity in heterogeneous anisotropic media. Gupta *et al.* (2010) formulated the influence of rigid boundary on the propagation of torsional wave in a homogeneous layer over a heterogeneous half space. The works done by Chattopadhyay *et al*. (2011, 2012), Gupta *et al.* (2013), Vishwakarma *et al*. (2012) on torsional wave propagation cannot be overlooked as their contributions are commendable. Kakar and Gupta (2013) studied torsional surface waves in a non-homogeneous isotropic layer over viscoelastic half-space. Kumari and Sharma (2014) discussed torsional waves in a viscoelastic layer over an inhomogeneous half space. Kakar (2014) analyzed the effect of gravity and nonhomogeneity on Rayleigh waves in higher-order elastic-viscoelastic half-space. Kakar and Gupta (2014) investigated the existence of Love waves in an intermediate heterogeneous layer placed in between homogeneous and inhomogeneous half-spaces using Green's function technique. Recently, Chattaraj *et al.* (2015) presented a note on torsional surface wave in dry sandy crust laid over an inhomogeneous half space. Dhua and Chattopadhyay (2015) studied torsional wave in an initially stressed layer sandwiched between two inhomogeneous media.

In this problem, we investigate the propagation of torsional wave in an in-homogeneous crustal layer over an inhomogeneous half space under the influence of rigid boundary and initial stress. The inhomogeneity of the crustal layer and elastic half space has been taken as $\mu=\mu_0(1+az)$, $\rho=\rho_0(1+bz)$, $P=P_0(1+cz)$ and $\mu=\mu_1\cosh^2(z/f)$, $\rho=\rho_1\cosh^2(z/f)$ where μ, ρ and P are the rigidity, mass density and initial stress respectively, a, b and c are constants having dimension that are inverse of length and f have the dimension of length. The effect of inhomogeneity parameter and initial stresses (tensile and compressional) on the propagation of torsional surface wave has been presented graphically by plotting the dispersion curves.

2. Formulation of the problem

Let H be the thickness of the initially stressed crustal layer with linear variation in rigidity and density placed over inhomogeneous half-space with hyperbolic variation in rigidity and density, where top of the layer is assumed to be rigid. We consider propagation of torsional waves which have only circumferential displacements (independent of azimuthal angle) in an in-homogeneous layer of finite thickness H. Let r-axis along the direction of torsional wave propagation and z-axis toward the interior of the elastic half-space. The variations of inhomogeneity in rigidity, density and initial stress in the crustal layer are taken as

$$\left.\begin{aligned}\mu&=\mu_0(1+az)\\ \rho&=\rho_0(1+bz)\\ P&=P_0(1+cz)\end{aligned}\right\}\tag{1}$$

and for elastic half- space inhomogeneity in rigidity and density are

$$\left.\begin{aligned}\mu&=\mu_1\cosh^2(z/f)\\ \rho&=\rho_1\cosh^2(z/f)\end{aligned}\right\}\tag{2}$$

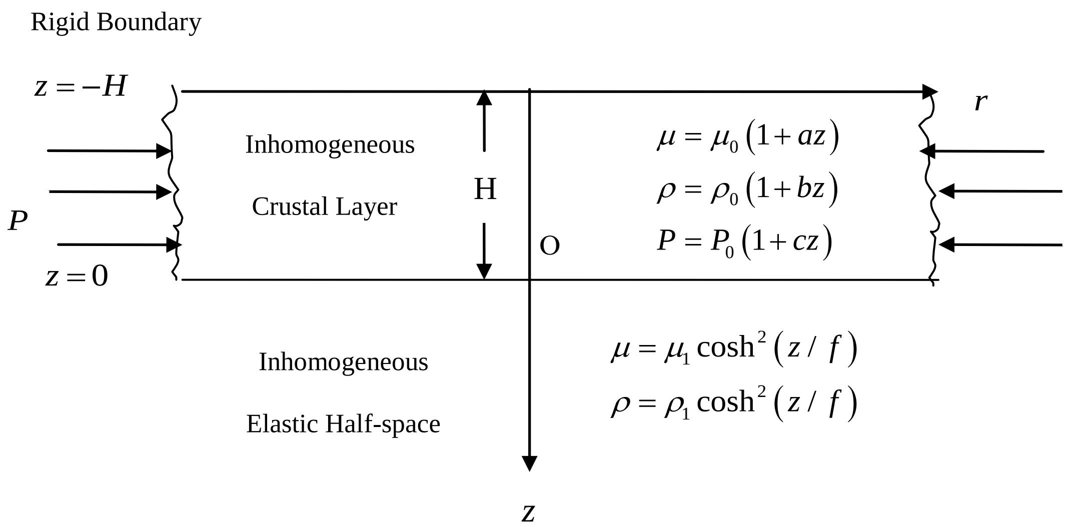

Fig. 1 Geometry of the problem

where $a>0, b>0, d>0, f>0$ and $\mu_0, \rho_0, \mu_1, \rho_1, P_0$ are constants.

3. Boundary conditions

The geometry of the problem leads to the following boundary conditions:
(i) At the free surface *z*=-H, when the upper boundary is assumed to be rigid then

$$v_0 = 0 \quad at \quad z = -\mathrm{H}. \tag{3a}$$

(ii) At the interface *z*=0, the continuity of the stress requires that

$$\mu_0 \frac{\partial v_0}{\partial z} = \mu_1 \frac{\partial v_1}{\partial z} \quad at \quad z = 0. \tag{3b}$$

(iii) The continuity of the displacement requires that

$$v_0 = v_1 \quad at \quad z = 0. \tag{3c}$$

(iv) Displacement is bounded as

$$\lim_{z\to\infty} v_1(z) = 0 \tag{3d}$$

where v_0 and v_1 are the displacement in the layer and the half-space respectively.

4. Solution of the problem

The dynamical equations of motion in cylindrical coordinate (r,θ,z) are (Love 1944)

$$\begin{aligned}&\frac{\partial s_{rr}}{\partial r}+\frac{1}{r}\frac{\partial s_{r\theta}}{\partial \theta}+\frac{\partial s_{rz}}{\partial z}+\frac{1}{r}(s_{rr}-s_{\theta\theta})+T_R=\rho\frac{\partial^2 u}{\partial t^2}\\&\frac{\partial s_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial s_{\theta\theta}}{\partial \theta}+\frac{\partial s_{\theta z}}{\partial z}+\frac{2s_{r\theta}}{r}+T_\theta=\rho\frac{\partial^2 v}{\partial t^2}\\&\frac{\partial s_{rz}}{\partial r}+\frac{1}{r}\frac{\partial s_{\theta z}}{\partial \theta}+\frac{\partial s_{zz}}{\partial z}+\frac{s_{rz}}{r}+T_Z=\rho\frac{\partial^2 w}{\partial t^2}\end{aligned}\tag{4}$$

where, $s_{rr}, s_{r\theta}, s_{rz}, s_{rr}.s_{\theta\theta}, s_{\theta z}, s_{zz}$ are the respective stress components, T_R, T_θ, T_Z are the respective body forces and u, v, w are the respective displacement components.

The stress-strain relation are given by

$$\begin{aligned}&s_{rr}=\lambda\Omega+2\mu e_{rr}, && s_{\theta\theta}=\lambda\Omega+2\mu e_{\theta\theta}\\&s_{zz}=\lambda\Omega+2\mu e_{zz}, && s_{r\theta}=2\mu e_{r\theta}\\&s_{rz}=2\mu e_{rz}, && s_{\theta z}=2\mu e_{\theta z}\end{aligned}\tag{5}$$

where λ and μ are Lame's constants, and $\Omega=\left(\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{u}{r}+\frac{\partial w}{\partial z}\right)$ denotes the dilatation.

The strain-displacement relations are

$$\begin{aligned}&e_{rr}=\frac{1}{2}\frac{\partial u}{\partial r}, e_{\theta\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{u}{r}\right), e_{zz}=\frac{1}{2}\frac{\partial w}{\partial z}\\&e_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right), e_{\theta z}=\frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \theta}\right), e_{zr}=\frac{1}{2}\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)\end{aligned}\tag{6}$$

The torsional wave is characterized by the displacements

$$u=0,\quad w=0,\quad v=v(r,z,t)\tag{7}$$

Now, considering Eq. (5)-(7), the dynamical equations of motion for torsional surface waves propagating in the radial direction under initial stress can be written as Biot (1965)

$$\frac{\partial \tau_{r\theta}}{\partial r}+\frac{\partial \tau_{z\theta}}{\partial z}+\frac{2\tau_{r\theta}}{r}-\frac{P(z)}{2}\frac{\partial^2 v}{\partial z^2}=\rho(z)\frac{\partial^2 v}{\partial t^2}\tag{8}$$

where $v(r,z,t)$ is the displacement along the θ (azimuthal) direction and r is the radial coordinates. The stress are related to the displacement component by

$$\tau_{r\theta}=\mu(z)\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right),\quad \tau_{z\theta}=\mu(z)\left(\frac{\partial v}{\partial z}\right)\tag{9}$$

Using Eq. (9), Eq. (8) takes the form

$$\mu(z)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)v+\frac{\partial}{\partial z}\left(G(z)\frac{\partial v}{\partial z}\right)=\rho(z)\frac{\partial^2 v}{\partial t^2}\tag{10}$$

where $G(z)=\mu(z)-\frac{P(z)}{2}$, $P(z)$ being the compressive initial stress in the medium along $P(z)$ direction Chattopathyay *et al.* (2013), Dua and Chattopadhayay (2015).

The solution of Eq. (10) is of the form

$$v=\mathrm{U}(z)\,\mathrm{J}_1(Kr)\exp(i\omega t) \tag{11}$$

where $\mathrm{U}(z)$ is the solution of the following equation

$$\frac{d^2\mathrm{U}(z)}{dz^2}+\frac{1}{G(z)}\frac{dG(z)}{dz}\frac{dU(z)}{dz}-\frac{\mu(z)K^2}{G(z)}\left(1-\frac{\beta^2}{{\beta_s}^2}\right)\mathrm{U}(z)=0 \tag{12}$$

In the above equation, $\beta=\frac{\omega}{K}$ is the torsional wave velocity, $\beta_s=\sqrt{\frac{\mu}{\rho}}$, ω is the angular frequency, K is the wave number and $J_1(Kr)$ is the first order Bessel function of first kind.

4.1 Solution for the crustal layer

The crustal layer is inhomogeneous isotropic with initial stress, so using the relation

$$\mathrm{U}(z)=\frac{\psi(z)}{G_1(1+\xi z)^{\frac{1}{2}}}$$

The Eq. (12) reduces to

$$\frac{d^2\psi(z)}{dz^2}+\frac{\psi(z)}{4\left[G_1(1+\xi z)\right]^2}\left(\frac{d\left[G_1(1+\xi z)\right]}{dz}\right)^2-\frac{K^2\mu_0(1+az)}{G_1(1+\xi z)}\left(1-\frac{\beta^2}{{\beta_0}^2}\frac{(1+bz)}{(1+az)}\right)\psi(z)=0, \tag{13}$$

We can rewrite the Eq. (13) as

$$\frac{d^2\psi(z)}{dz^2}+\left[\frac{\xi^2}{4(1+\xi z)^2}-K_1^2\left\{\frac{(1+az)}{(1+\xi z)}-\frac{\beta^2}{{\beta_0}^2}\frac{(1+cz)}{(1+\xi z)}\right\}\right]\psi(z)=0, \tag{14}$$

where $\beta_0=\sqrt{\frac{\mu_0}{\rho_0}}$, $K_1^2=\frac{K\mu_0}{G_1}$, $G_1=\left(\mu_0-\frac{P_0}{2}\right)$, $\xi=\frac{(a-\zeta b)}{(1-\zeta)}$ and $\zeta=\frac{P_0}{2\mu_0}$.

Introducing $\psi(z)=\phi(\eta)$ in the Eq. (14), where $\xi=\frac{2MK_1(1+\xi z)}{\xi}$, we have

$$\phi''(\eta)+\left(\frac{1}{4\eta^2}-\frac{1}{4}+\frac{R}{\eta}\right)\phi(\eta)=0, \tag{15}$$

where $R=\frac{K_1}{2M\xi}\left[\left(\frac{\beta^2}{\beta_0{}^2}-1\right)+M^2\right]$ and $M=\frac{1}{\xi}\left[\left(a-\frac{\beta^2}{\beta_0{}^2}c\right)\right]^{\frac{1}{2}}$ are dimensionless quantities.

Eq. (15) is the well known Whittaker's equation [Whittaker and Watson (1990)].

The solution of Whittaker's Eq. (15) is given by

$$\phi(\xi)=AW_{\frac{R}{2},0}(\eta)+BW_{-\frac{R}{2},0}(-\eta) \tag{16}$$

where A and B are arbitrary constants and $W_{\frac{R}{2},0}(\eta), W_{-\frac{R}{2},0}(-\eta)$ are the Whittaker function.

Hence the displacement for the torsional wave in the crustal layer is

$$\psi(z)=AW_{\frac{R}{2},0}\left[\frac{2MK_1}{\xi}(1+\xi\,\mathrm{z})\right]+BW_{-\frac{R}{2},0}\left[-\frac{2MK_1}{\xi}(1+\xi\,\mathrm{z})\right]$$

So

$$\mathrm{U}(z)=\frac{AW_{\frac{R}{2},0}\left[\frac{2MK_1}{\xi}(1+\xi\,\mathrm{z})\right]+BW_{-\frac{R}{2},0}\left[-\frac{2MK_1}{\xi}(1+\xi\,\mathrm{z})\right]}{G_1(1+\xi z)^{\frac{1}{2}}}$$

and

$$v_0=\left(\frac{AW_{\frac{R}{2},0}\left[\frac{2MK_1}{\xi}(1+\xi\,\mathrm{z})\right]+BW_{-\frac{R}{2},0}\left[-\frac{2MK_1}{\xi}(1+\xi\,\mathrm{z})\right]}{\sqrt{G_1}(1+\xi z)^{\frac{1}{2}}}\right)J_1(Kr)\exp(i\omega t). \tag{17}$$

4.2 Solution for the lower in-homogeneous elastic half space

Putting $\mathrm{U}=\frac{\mathrm{U}_1}{\sqrt{\mu}}$ in Eq. (12) without taking initial stress and using Eq. (2), we get

$$U_1''-\delta_1^2U_1=0 \tag{18}$$

where $\delta_1=K\left(1+\frac{1}{f^2K^2}-\frac{\beta^2}{\beta_1^2}\right)^{\frac{1}{2}}$.

The dispersion relation for torsional waves can be obtained by using boundary conditions Eq. (3). Therefore, the displacement for the torsional wave in the in-homogeneous half-space using boundary condition (3d) becomes

$$v_1 = \frac{Ce^{-\delta_1 z}}{\sqrt{\mu_1}\cosh\left(\frac{z}{f}\right)} J_1(Kr) e^{i\omega t} \tag{19}$$

Now using the boundary conditions (3b), (3c) and (3a), we get

$$\frac{1}{\sqrt{G_1}}\left[A\left\{-\frac{\xi}{2}W_{\frac{R}{2},0}\left[\frac{2MK_1}{\xi}\right]+2MK_1W'_{\frac{R}{2},0}\left[\frac{2MK_1}{\xi}\right]\right\}-B\left\{\frac{\xi}{2}W_{-\frac{R}{2},0}\left[-\frac{2MK_1}{\xi}+2MK_1W'_{\frac{R}{2},0}\left[-\frac{2MK_1}{\xi}\right]\right]\right\}\right]$$
$$-C\left(\frac{\mu_1}{\mu_0}\right)\left(1+\frac{1}{f^2K^2}-\frac{\beta^2}{\beta_1^2}\right)^{-\frac{1}{2}}=0 \tag{20}$$

$$\frac{1}{\sqrt{G_1}}\left[A\left\{W_{\frac{R}{2},0}\left[\frac{2MK_1}{\xi}\right]+BW_{-\frac{R}{2},0}\left[-\frac{2MK_1}{\xi}\right]\right\}\right]-C=0 \tag{21}$$

$$\frac{1}{\sqrt{G_1(1-\xi\mathrm{H})}}\left\{AW_{\frac{R}{2},0}\left[\frac{2MK_1}{\xi}(1-\xi\mathrm{H})\right]+BW_{-\frac{R}{2},0}\left[-\frac{2MK_1}{\xi}(1-\xi\mathrm{H})\right]\right\}-C=0 \tag{22}$$

From Eq. (20), (21) and (22) to eliminate A, B and C we have

$$\begin{vmatrix} \frac{1}{\sqrt{G_1}}W_{\frac{R}{2},0}\left[\frac{2MK_1}{\xi}\right] & \frac{1}{\sqrt{G_1}}W_{-\frac{R}{2},0}\left[-\frac{2MK_1}{\xi}\right] & -1 \\ \frac{1}{\sqrt{G_1}}\left\{-\frac{\xi}{2}W_{\frac{R}{2},0}\left[\frac{2MK_1}{\xi}\right]+2MK_1W'_{\frac{R}{2},0}\left[\frac{2MK_1}{\xi}\right]\right\} & \frac{1}{\sqrt{G_1}}\left\{\frac{\xi}{2}W_{-\frac{R}{2},0}\left[-\frac{2MK_1}{\xi}\right]+2MK_1W'_{\frac{R}{2},0}\left[-\frac{2MK_1}{\xi}\right]\right\} & -\frac{\mu_1}{\mu_0}\left(1+\frac{1}{f^2K^2}-\frac{\beta^2}{\beta_1^2}\right)^{-\frac{1}{2}} \\ \frac{W_{\frac{R}{2},0}}{\sqrt{G_1(1-\xi\mathrm{H})}}\left[\frac{2MK_1}{\xi}(1-\xi\mathrm{H})\right] & \frac{W_{-\frac{R}{2},0}}{\sqrt{G_1(1-\xi\mathrm{H})}}\left[-\frac{2MK_1}{\xi}(1-\xi\mathrm{H})\right] & -1 \end{vmatrix}=0. \tag{23}$$

Set $\eta' = \frac{1}{\sqrt{G_1}}\left[\frac{2MK_1}{\xi}\right]$ and $\eta = \frac{1}{\sqrt{G_1(1-\xi\mathrm{H})}}\frac{2MK_1}{\xi}(1-\xi\mathrm{H})$

The expansion of Whittaker function up to linear term is taken as the form

$$W_{R,0}(\eta) = e^{-\eta/2}\eta^{\frac{R}{2}}\left\{1-\frac{(R/2-0.5)^2}{1!\eta}\right\} \tag{24}$$

Now expanding the above determinant we get

$$\frac{D_4e^{(MKH)}+D_3e^{(-MKH)}}{D_2e^{(MKH)}-D_1e^{(-MKH)}}+\frac{a}{2K\sqrt{1-\frac{\beta^2}{\beta_0^2}\left(\frac{b}{a}\right)}}=\frac{\mu_1}{\mu_0}\frac{\sqrt{1+\frac{1}{f^2K^2}-\frac{\beta^2}{\beta_1^2}}}{\sqrt{1-\frac{\beta^2}{\beta_0^2}\left(\frac{b}{a}\right)}} \tag{25}$$

where

$$D_1 = \left(\frac{\eta'}{\eta}\right)^{\frac{R}{2}} (-1)^{-\frac{R}{2}} \left(1 + \frac{\left(-\frac{R}{2} - 0.5\right)^2}{\eta}\right)\left(1 - \frac{\left(\frac{R}{2} - 0.5\right)^2}{\eta'}\right)$$

$$D_2 = \left(\frac{\eta}{\eta'}\right)^{\frac{R}{2}} (-1)^{-\frac{R}{2}} \left(1 - \frac{\left(\frac{R}{2} - 0.5\right)^2}{\eta}\right)\left(1 + \frac{\left(-\frac{R}{2} - 0.5\right)^2}{\eta'}\right)$$

$$D_3 = \left(\frac{\eta'}{\eta}\right)^{\frac{R}{2}} (-1)^{-\frac{R}{2}} \left(1 + \frac{\left(-\frac{R}{2} - 0.5\right)^2}{\eta}\right)\left(-1 + \frac{R}{\eta'} + \frac{1}{\eta'}\left(\frac{R}{2} - 0.5\right)^2 - R\frac{\left(\frac{R}{2} - 0.5\right)^2}{\eta'^2} + \frac{\left(\frac{R}{2} - 0.5\right)^2}{\eta'^2}\right)$$

$$D_4 = \left(\frac{\eta}{\eta'}\right)^{\frac{R}{2}} (-1)^{-\frac{R}{2}} \left(1 - \frac{\left(\frac{R}{2} - 0.5\right)^2}{\eta}\right)\left(-1 + \frac{R}{\eta'} - \frac{1}{\eta'}\left(-\frac{R}{2} - 0.5\right)^2 + R\frac{\left(-\frac{R}{2} - 0.5\right)^2}{\eta'^2} + \frac{\left(-\frac{R}{2} - 0.5\right)^2}{\eta'^2}\right)$$

Eq. (25) is the dispersion equation of torsional wave in an in-homogeneous prestressed crustal layer over an in-homogeneous half space when the upper boundary plane is assumed to rigid.

Special cases

i. For torsional wave in an in-homogeneous prestressed layer over an isotropic homogeneous half space when the upper boundary plane is assumed to be rigid, in that case $f \to \infty$, then Eq. (25) reduces to

$$\frac{D_4 e^{(MKH)} + D_3 e^{(-MKH)}}{D_2 e^{(MKH)} - D_1 e^{(-MKH)}} + \frac{a}{2K\sqrt{1 - \frac{\beta^2}{\beta_0^{\,2}}\left(\frac{b}{a}\right)}} = \frac{\mu_1}{\mu_0}\frac{\sqrt{1 - \frac{\beta^2}{\beta_1^{\,2}}}}{\sqrt{1 - \frac{\beta^2}{\beta_0^{\,2}}\left(\frac{b}{a}\right)}} \tag{26}$$

ii. For torsional wave in a homogeneous layer over an isotropic homogeneous half space when the upper boundary plane is assumed to be rigid, in that case $P \to 0, a \to 0, b \to 0, f \to \infty$, then Eq. (25) reduces to

$$\cot\left(\mathrm{KH}\sqrt{\frac{\beta^2}{\beta_0^{\,2}} - 1}\right) = \frac{\mu_1}{\mu_0}\frac{\sqrt{1 - \frac{\beta^2}{\beta_1}}}{\sqrt{\frac{\beta^2}{\beta_0^{\,2}} - 1}} \tag{27}$$

iii. For torsional wave in a homogeneous layer over an inhomogeneous half space when the upper boundary plane is assumed to be rigid, in that case $P \to 0, a \to 0, b \to 0,$ then Eq. (25) reduces to

$$\cot\left(\mathrm{KH}\sqrt{1-\frac{\beta^2}{\beta_0{}^2}}\right)=\frac{\mu_1}{\mu_0}\frac{\sqrt{1+\dfrac{1}{f^2K^2}-\dfrac{\beta^2}{\beta_1{}^2}}}{\sqrt{\dfrac{\beta^2}{\beta_0{}^2}-1}} \tag{28}$$

iv. For torsional wave in a homogeneous layer over an isotropic homogeneous half space when the upper boundary plane is assumed to be free, in that case $P \to 0,\ a \to 0, b \to 0,\ f \to \infty$, then Eq. (25) reduces to

$$\tan\left(KH\sqrt{\frac{\beta^2}{\beta_0^2}-1}\right)=\frac{\mu_1}{\mu_0}\frac{\sqrt{\left(1-\dfrac{\beta^2}{\beta_1^2}\right)}}{\sqrt{\left(\dfrac{\beta^2}{\beta_0^2}-1\right)}} \tag{29}$$

Eq. (29) is the classical dispersion equation of Love waves given by Love (1911) and Ewing *et al.* (1957), which validates our solution.

6. Numerical analysis

To show the effect of inhomogeneity parameters $l=\dfrac{a}{K}$ and $m=\dfrac{b}{K}$ on nature of torsional wave motion, we have plotted non-dimensional phase velocity β/β_0 against dimensionless wave number *KH* on the propagation of torsional wave in the crustal layer by using MATLAB software. Figs. 2-3 are plotted for Eq. (25) and Eq. (27) by taking parameters in Table 1, Gubbins (1990).

In Figs. 2 and 3 curves have been plotted with vertical axis as dimensionless phase velocity non-dimensional phase velocity β/β_0 against dimensionless wave number *KH*. In both the Figs. 2 and 3, the curve 1 denotes the classical case of Love wave. However curves 2,3,4,5 of figure 2 are drawn for the various values of in-homogeneity parameter $l=\dfrac{a}{K}$ keeping $m=\dfrac{b}{K}$ and $\zeta=\dfrac{P_0}{2\mu_0}$ constant and curves 2,3,4,5 of Fig. 3 are drawn for the various values of in-homogeneity parameter

Table 1 Material parameters

Layer	Rigidity	Density
Inhomogeneous Crustal Layer	$\mu_0 = 6.34\times10^{10}\,N/m^2$	$\rho_0 = 3364\,Kg/m^3$
Inhomogeneous elastic Half-space	$\mu_1 = 11.77\times10^{10}\,N/m^2$	$\rho_1 = 4148\,Kg/m^3$

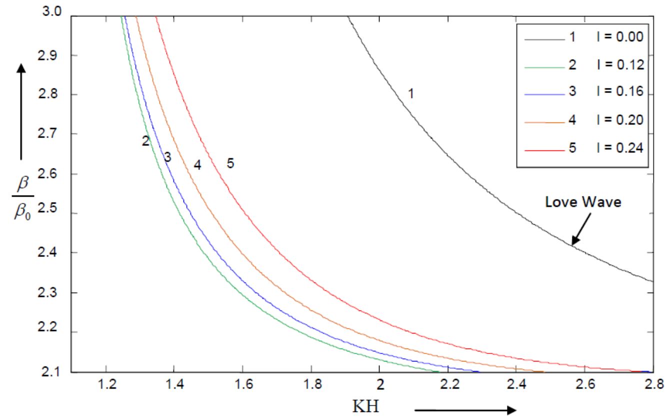

Fig. 2 Variation of dimensionless phase velocity against dimensionless wave number when m=0.8, ζ=0.5 and fK=0.5

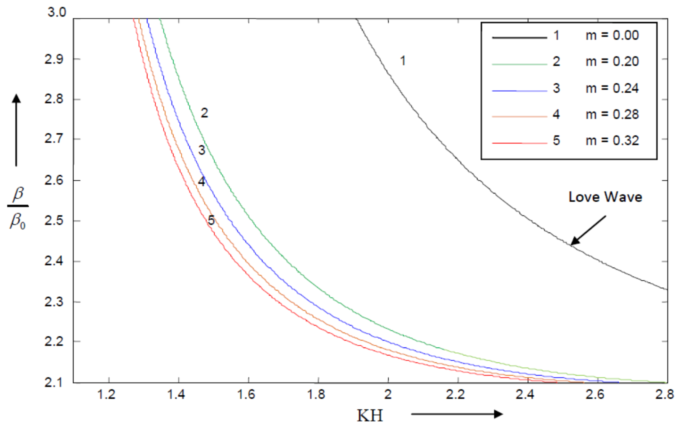

Fig. 3 Variation of dimensionless phase velocity against dimensionless wave number for l=0.4, ζ=0.5 and fK=0.5

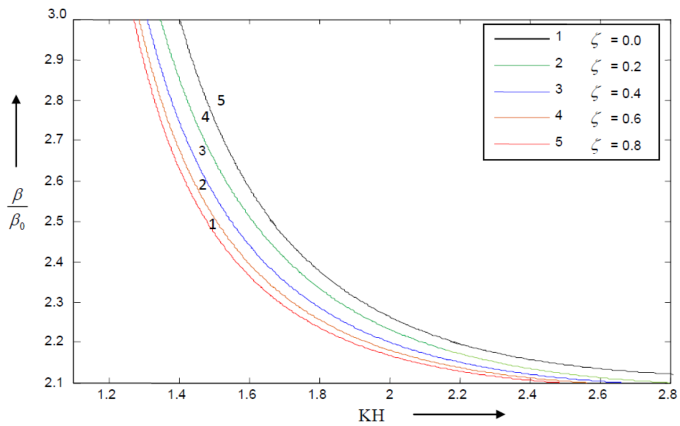

Fig. 4 Variation of dimensionless phase velocity against dimensionless wave number when l=0.4, m=0.8 and fK=0.5

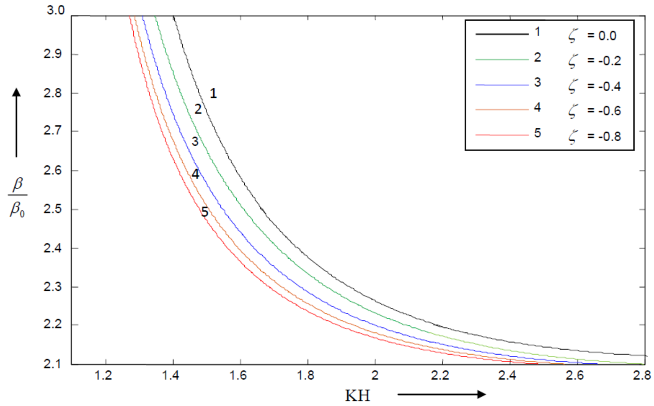

Fig. 5 Variation of dimensionless phase velocity against dimensionless wave number when l=0.4, m=0.8 and fK=0.5

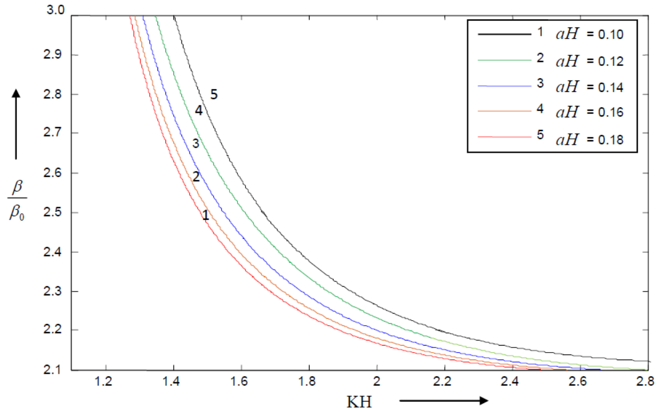

Fig. 6 Variation of dimensionless phase velocity against dimensionless wave number under the effect of inhomogeneity associated with rigidity the layer

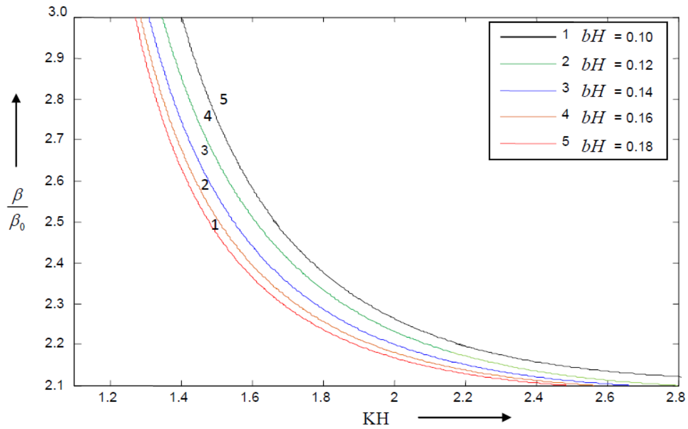

Fig. 7 Variation of dimensionless phase velocity against dimensionless wave number under the effect of inhomogeneity associated with density the layer

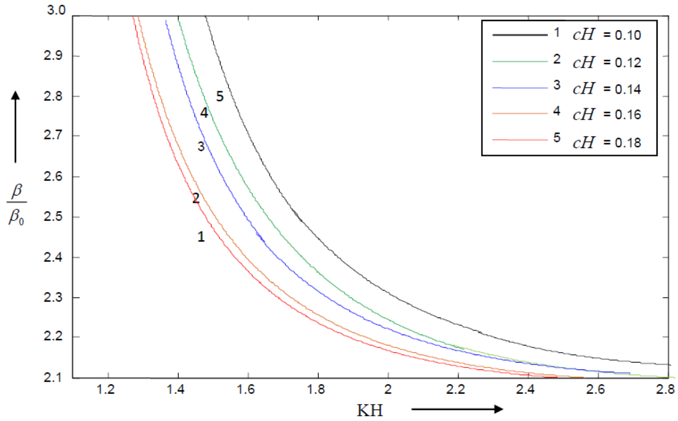

Fig. 8 Variation of dimensionless phase velocity against dimensionless wave number under the effect of inhomogeneity associated with initial stress the layer

$m=\frac{b}{K}$ keeping $l=\frac{a}{K}$ and $\zeta=\frac{P_0}{2\mu_0}$ constant. Fig. 2 signifies the dispersion curve of the torsional surface wave under the effect of in-homogeneity parameter $l=\frac{a}{K}$ involved in the rigidity of the crustal layer. From this figure, it is clear that speed of torsional wave decreases with the increase of wave number. Also from curve no 2 to 5 we observed that the velocity of the torsional wave is directly proportional to the rigidity of the medium. Fig. 3 represents the dispersion curve of the torsional wave under the effect of in-homogeneity parameter $m=\frac{b}{K}$ involved in the density of the crustal layer. The speed of torsional surface wave decreases rapidly with the increase of wave number. Also from curve no 2 to 5 we observed that the velocity of the torsional wave is inversely proportional to the density of the medium. Fig. 4 depicting the dispersion curve of the torsional wave, shows the effect of parameter $\zeta=\frac{P_0}{2\mu_0}$ designating the stress of the crustal layer. The speed of the torsional surface wave increases rapidly with the increase of wave number. Also from curves 2-5, we observe that the velocity of the torsional wave is directly proportional to the tensile stresses of the medium. In Fig. 5 curves have been plotted with vertical axis as dimensionless phase velocity non-dimensional phase velocity β/β_0 against dimensionless wave number KH for compressional stresses. Also from curves 2-5, in Fig. 5, we observe that the velocity of the torsional wave is inversely proportional to the compressional stresses of the medium.

In Figs. 6 and 7curves have been plotted with vertical axis as dimensionless phase velocity non-dimensional phase velocity β/β_0 against dimensionless wave number KH under the effect of inhomogeneity associated with rigidity and density of the layer, respectively. Fig. 6 signifies the dispersion curve of the torsional surface wave under the effect of in-homogeneity parameter aH associated with rigidity of the crustal layer. From this figure, it is clear that speed of torsional wave decreases with the increase of wave number. Also from curve no 2 to 5 we observed that the velocity of the torsional wave is directly proportional to the rigidity of the medium. Fig. 7 represents the dispersion curve of the torsional wave under the effect of in-homogeneity parameter bH involved in the density of the crustal layer. The speed of torsional surface wave decreases rapidly with the increase of wave number. Also from curve no 2 to 5 we observed that the velocity of the torsional wave is inversely proportional to the density of the medium. Fig. 8 represents the dispersion curve of the torsional wave under the effect of in-homogeneity parameter cH involved in the initial stress of the crustal layer. The speed of torsional surface wave increases rapidly with the increase of wave number. Also from curve no 2 to 5 we observed that the velocity of the torsional wave is directly proportional to the density of the medium.

7. Conclusions

We have studied the propagation of torsional wave in an inhomogeneous initially stressed crustal layer over an inhomogeneous half space when the upper boundary plane is assumed to be rigid. We have employed Whittaker's function to find the frequency equation of torsional surface wave. Close form solutions for the displacement in the crustal layer have been derived. In a particular case, the dispersion equation coincides with the well-known classical equation of Love wave when the crustal layer and lower half-space are homogeneous for upper free space, and hence it validates the solution of the problem. From above numerical analysis, it may be conclude that:

a. The velocity of the torsional wave is inversely proportional to the inhomogeneity factor aH of the medium.

b. The inhomogeneity factor $\frac{a}{K}$ has a prominent effect on torsional wave propagation. The velocity of the torsional wave is directly proportional to the rigidity of the medium.

c. The velocity of the torsional wave is inversely proportional to inhomogeneity factor bH of the medium.

d. The velocity of the torsional wave is directly proportional to inhomogeneity factor cH of the medium.

e. The inhomogeneity factor $\frac{b}{K}$ has a prominent effect on torsional wave propagation. The velocity of the torsional wave is inversely proportional to the density of the medium.

f. Phase velocity $\frac{\beta}{\beta_0}$ (non-dimensional) of torsional wave decreases with increase of wave number kH (non-dimensional).

g. The velocity of the torsional wave is inversely proportional to the compressional stresses $\zeta = -\frac{P_0}{2\mu_0}$ of the medium and directly proportional tensile stresses $\zeta = +\frac{P_0}{2\mu_0}$.

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