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Accurate periodic solution for non-linear vibration of dynamical equations

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Abstract. In this paper we consider three different cases and we apply Variational Approach (VA) to solve the non-natural vibrations and oscillations. The method variational approach does not demand small perturbation and with only one iteration can lead to high accurate solution of the problem. Some patterns are presented for these three different cease to show the accuracy and effectiveness of the method. The results are compared with numerical solution using Runge-kutta's algorithm and another approximate method using energy balance method. It has been established that the variational approach can be an effective mathematical tool for solving conservative nonlinear dynamical equations.

Keywords: variational approach; nonlinear oscillators; mathematical pendulum; runge-Kutta's algorithm

1. Introduction

Nonlinear oscillation in engineering and applied mathematics has been a topic to intensive research for many years. Nonlinear oscillator models have been widely considered in physics and engineering. Many authors used various analytical methods for solving nonlinear oscillation systems. The traditional perturbation methods have many shortcomings, and they are not valid for strongly nonlinear equations. To handle the nonlinear problems, many new mathematical methods have appeared in open literatures recently, for example: ; Homotopy perturbation method (Shaban *et al*. 2010; Bayat 2013a),Hamiltonian approach (Bayat *et al*. 2011a; 2012a; 2013a, b; 2014a, b), energy balance method (He 2002; Bayat *et al*. 2011b, Pakar *et al*. 2011a, b; Mehdipour 2010), Variational iteration method (Dehghan 2010; Pakar *et al*. 2012), Amplitude frequency formulation (Bayat 2011c; 2012b; Pakar *et al*. 2013a; He2008), max-min approach (Shen *et al*. 2009; Zeng *et al*. 2009), Variational approach method (He2007; Bayat *et al*. 2012c; 2013c; 2014c; Pakar *et al*. 2012b), and the other analytical and numerical (Xu2009; Alicia *et al*. 2010; Bor-Lih *et al*. 2009; Wu2011; Odibat *et al*. 2008). In this paper, we will show how to solve the problems of nonlinear oscillators by a new variational approach proposed by He (2007), which is an easy, effective and convenient mathematical tool and analytical formulas for the period and periodic solution.

In this paper, at first, we describe basic idea of he's variational approach method and then, the VA will be applied to the three strong dynamical equations. Finally, the frequency of the oscillator

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obtained by He's energy balance method is given to demonstrate the validity of the proposed method (VA). The comparison shows the efficiency of these methods. As we can see, the results are presented in this paper reveal that variational approachis very effective and convenient for nonlinear oscillators. Procedure of numerical solution is also presented in Appendix A.

2. Basic idea of He's variational approach

He (2007) suggested a variational approach which is different from the known variational methods in open literature. Hereby we give a brief introduction of the method:

$$\ddot{u} + f\left(u\right) = 0 \tag{1}$$

Its variational principle can be easily established using the semi-inverse method

$$J\left(u\right) = \int_0^{T/4} \left(-\frac{1}{2}\dot{u}^2 + F\left(u\right)\right) dt \tag{2}$$

where Tis period of the nonlinear oscillator, $\partial F / \partial u = f$. Assume that its solution can be expressed as

$$u\left(t\right) = A\cos\left(\omega t\right) \tag{3}$$

where A and ω are the amplitude and frequency of the oscillator, respectively. Substituting Eq. (3) into Eq. (2) results in:

$$\begin{aligned} J\left(A,\omega\right) &= \int_0^{T/4} \left(-\frac{1}{2}A^2\omega^2 \sin^2 \omega t + F\left(A\cos\omega t\right)\right) dt \\ &= \frac{1}{\omega}\int_0^{\pi/2} \left(-\frac{1}{2}A^2\omega^2 \sin^2 t + F\left(A\cos t\right)\right) dt \\ &= -\frac{1}{2}A^2\omega\int_0^{\pi/2} \sin^2 t\, dt + \frac{1}{\omega}\int_0^{\pi/2} F\left(A\cos t\right) dt \end{aligned} \tag{4}$$

Applying the Ritz method, we require:

$$\frac{\partial J}{\partial A} = 0 \tag{5}$$

$$\frac{\partial J}{\partial \omega} = 0 \tag{6}$$

But with a careful inspection, for most cases He find that

$$\frac{\partial J}{\partial \omega} = -\frac{1}{2}A^2\int_0^{\pi/2} \sin^2 t\, dt - \frac{1}{\omega^2}\int_0^{\pi/2} F\left(A\cos t\right) dt < 0 \tag{7}$$

Thus, He modify conditions Eq. (5) and Eq. (6) into a simpler form:

$$\frac{\partial J}{\partial \omega} = 0 \tag{8}$$

From which the relationship between the amplitude and frequency of the oscillator can be obtained.

3. Applications

In this section, we will present three examples to illustrate the applicability, accuracy and effectiveness of the proposed approach.

3.1 Example 1

We consider the rigid frame (Fig. 1) is forced to rotate at the fixed rate while the frame rotate, the governing equation of the simple pendulum oscillates is (Nayfeh 1981);

$$\ddot{\theta} - \Omega^2 cos(\theta) sin(\theta) + \frac{g}{r} sin(\theta) = 0 \tag{9}$$

with the boundary conditions of:

$$\theta(0) = A, \qquad \dot{\theta}(0) = 0 \tag{10}$$

In order to apply the variational approach method to solve the above problem, the approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$ and $\sin\theta \approx \theta - \frac{1}{6}\theta^3$ is used.

Its variational formulation can be readily obtained as follows:

$$J(\theta) = \int_0^{T/4} \left(-\frac{1}{2}\dot{\theta}^2 - \frac{1}{2}\Omega^2 \theta^2 + \frac{1}{6}\Omega^2\theta^4 - \frac{1}{48}\Omega^2\theta^6 + \frac{1}{1152}\Omega^2\theta^8 + \frac{1}{2}\frac{g}{r}\theta^2 - \frac{1}{24}\frac{g}{r}\theta^4 \right) dt \tag{11}$$

Choosing the trial function $\theta(t) = A\cos(\omega t)$ into Eq. (11) we obtain

$$J(A,\omega) = \int_0^{T/4} \begin{pmatrix} -\frac{1}{2}\left(\mathrm{A}\,\omega \sin(\omega \mathrm{t})\right)^2 - \frac{1}{2}\Omega^2\left(A\cos(\omega t)\right)^2 + \frac{1}{6}\Omega^2\left(A\cos(\omega t)\right)^4 \\ -\frac{1}{48}\Omega^2\left(A\cos(\omega t)\right)^6 + \frac{1}{1152}\Omega^2\left(A\cos(\omega t)\right)^8 \\ +\frac{1}{2}\frac{g}{r}\left(A\cos(\omega t)\right)^2 - \frac{1}{24}\frac{g}{r}\left(A\cos(\omega t)\right)^4 \end{pmatrix} dt \tag{12}$$

The stationary condition with respect to *A* reads:

$$\frac{\partial J}{\partial A} = \int_0^{T/4} \begin{pmatrix} -A\,\omega^2 sin^2(\omega t) - \Omega^2 A cos^2(\omega t) + \frac{2}{3}\Omega^2 A^3 cos^4(\omega t) - \frac{1}{8}\Omega^2 A^5 cos^6(\omega t) \\ +\frac{1}{144}\Omega^2 A^7 cos^8(\omega t) + \frac{g}{r} A\, cos^2(\omega t) - \frac{1}{6}\frac{g}{r} A^3 cos^4(\omega t) \end{pmatrix} dt = 0 \tag{13}$$

or

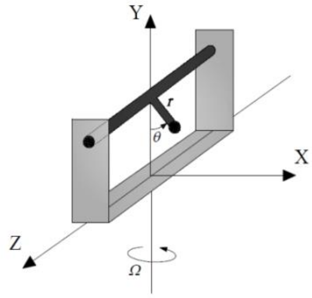

Fig. 1 Simple pendulum attached to rotating rigid frame

$$\frac{\partial J}{\partial A}=\int_0^{\pi/2}\begin{pmatrix} -A\,\omega^2 sin^2 t-\Omega^2 A\cos^2 t+\frac{2}{3}\Omega^2 A^3 cos^4 t-\frac{1}{8}\Omega^2 A^5 cos^6 t \\ +\frac{1}{144}\Omega^2 A^7 cos^8 t+\frac{g}{r}A\cos^2 t-\frac{1}{6}\frac{g}{r}A^3 cos^4 t \end{pmatrix} dt=0 \tag{14}$$

Then we have;

$$\omega^2=\frac{\int_0^{\pi/2}\begin{pmatrix} -\Omega^2 A\cos^2 t+\frac{2}{3}\Omega^2 A^3 cos^4 t-\frac{1}{8}\Omega^2 A^5 cos^6 t \\ +\frac{1}{144}\Omega^2 A^7 cos^8 t+\frac{g}{r}A\cos^2 t-\frac{1}{6}\frac{g}{r}A^3 cos^4 t \end{pmatrix} dt}{A\int_0^{\pi/2} sin^2 t\ dt} \tag{15}$$

Solving Eq. (15), according to ω, we have:

$$\omega_{VA}=\frac{1}{96}\sqrt{-9216\Omega^2+4608\Omega^2 A^2-720\Omega^2 A^4+35\Omega^2 A^6+9216\frac{g}{r}-1152\frac{g}{r}A^2} \tag{16}$$

Hence, the approximate solution can be readily obtained:

$$\theta(t)=A\cos\left(\frac{1}{96}\sqrt{-9216\Omega^2+4608\Omega^2 A^2-720\Omega^2 A^4+35\Omega^2 A^6+9216\frac{g}{r}-1152\frac{g}{r}A^2}\ t\right) \tag{17}$$

For comparison of the approximate solution, frequency obtained from solution of nonlinear equation with the energy balance method is:

$$\omega_{EBM}=\frac{\sqrt{6}}{96}\sqrt{-1536\Omega^2+768\Omega^2 A^2-112\Omega^2 A^4+5\Omega^2 A^6+1536\frac{g}{r}-192\frac{g}{r}A^2} \tag{18}$$

The numerical solution by with 4th order Runge-Kutta method (Appendix A) for nonlinear equation is:

$$\begin{aligned} &\dot{\theta}=\mathrm{y} && \theta(0)=\mathrm{A} \\ &\dot{y}=\Omega^2 cos(\theta)sin(\theta)-\frac{g}{r}sin(\theta) && y(0)=0 \end{aligned} \tag{19}$$

3.2 Example 2

Nonlinear approximate equations for eccentrically reinforced cylindrical shell are obtained by considering a simplified boundary value problem by (Andrianov 2004) .The governing equation of simply supported shell can be expressed as:

$$\ddot{u}+au\left(\dot{u}^{2}+u\,\ddot{u}\right)+bu+cu^{3}+du^{5}=0 \tag{20}$$

with the boundary conditions of:

$$u(0)=A, \qquad \dot{u}(0)=0 \tag{21}$$

Its variational formulation can be readily obtained as follows:

$$J(u)=\int_0^{T/4}\left(-\frac{1}{2}\dot{u}^2-\frac{1}{2}a\dot{u}^2u^2+\frac{1}{2}bu^2+\frac{1}{4}cu^4+\frac{1}{6}du^6\right)dt \tag{22}$$

Choosing the trial function $\theta(t)=A\cos(\omega t)$ into Eq. (22) we obtain

$$J(A,\omega)=\int_0^{T/4}\left(\begin{array}{l}-\frac{1}{2}\left(A\,\omega\sin(\omega t)\right)^2-\frac{1}{2}a\left(A\,\omega\sin(\omega t)\right)^2\left(A\cos(\omega t)\right)^2\\+\frac{1}{2}b\left(A\cos(\omega t)\right)^2+\frac{1}{4}c\left(A\cos(\omega t)\right)^4+\frac{1}{6}d\left(A\cos(\omega t)\right)^6\end{array}\right)dt \tag{23}$$

The stationary condition with respect to A reads:

$$\frac{\partial J}{\partial A}=\int_0^{T/4}\left(\begin{array}{l}-A\,\omega^2 sin^2(\omega t)-2a\,\omega^2A^3 sin^2(\omega t)cos^2(\omega t)+b\,A\,cos^2(\omega t)\\+c\,A^3cos^4(\omega t)+d\,A^5cos^6(\omega t)\end{array}\right)dt=0 \tag{24}$$

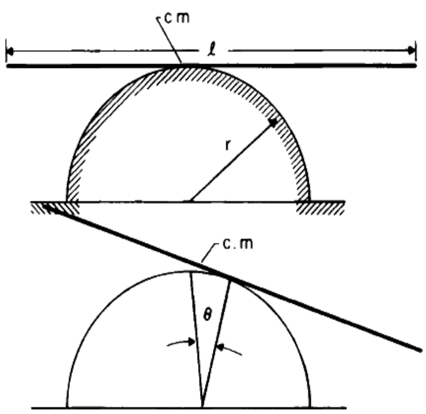

Fig. 2 Rigid rod rocks on circular surface (Nayfeh 1995)

or

$$\frac{\partial J}{\partial A}=\int_0^{\pi/2}\left(-A\,\omega^2 sin^2 t-2a\,\omega^2 A^3 sin^2 t\,cos^2 t+b\,A\,cos^2 t+c\,A^3 cos^4 t+d\,A^5 cos^6 t\right)dt=0 \tag{25}$$

Then we have;

$$\omega^2=\frac{\int_0^{\pi/2}\left(b\,A\,cos^2 t+c\,A^3 cos^4 t+d\,A^5 cos^6 t\right)dt}{\int_0^{\pi/2}\left(A\,sin^2 t+2a\,A^3 sin^2 t\,cos^2 t\right)dt} \tag{26}$$

Solving Eq. (26), according to ω, we have:

$$\omega_{VA}=\frac{1}{2}\sqrt{\frac{8b+6cA^2+5d\,A^4}{aA^2+2}} \tag{27}$$

Hence, the approximate solution can be readily obtained:

$$\theta(t)=A\cos\left(\frac{1}{2}\sqrt{\frac{8b+6cA^2+5d\,A^4}{aA^2+2}}\;t\right) \tag{28}$$

For comparison of the approximate solution, frequency obtained from solution of nonlinear equation with the energy balance method is:

$$\omega_{EBM}=\frac{1}{\sqrt{6}}\sqrt{\frac{12b+9cA^2+7dA^4}{a\,A^2+2}} \tag{29}$$

The numerical solution by with 4th order Runge-Kutta (Appendix A) method for nonlinear equation is:

$$\begin{aligned}&\dot{u}=y && u(0)=A\\ &\dot{y}=-\frac{1}{1+au^2}\left(au\,\dot{u}^2+bu+cu^3+d\,u^5\right) && y(0)=0\end{aligned} \tag{30}$$

3.3 Example 3

The governing equation of the rigid rod (Fig. 2) rocks back and forth on the circular surface without slipping, is (Nayfeh1981):

$$\left(\tfrac{1}{12}l^2+r^2\theta^2\right)\ddot{\theta}+r^2\theta\dot{\theta}^2+r\,g\,\theta cos(\theta)=0 \tag{31}$$

with the boundary conditions of:

$$\theta(0) = A, \qquad \dot{\theta}(0) = 0 \tag{32}$$

In order to apply the variational approach method to solve the above problem, the approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$ is used. Its variational formulation is:

$$J(\theta) = \int_0^{T/4} \left(-\frac{1}{24} l^2 \dot{\theta}^2 - \frac{1}{2} r^2 \theta^2 \dot{\theta}^2 + \frac{1}{2} r\, g\, \theta^2 - \frac{1}{8} r\, g\, \theta^4 + \frac{1}{144} g\, r\, \theta^6 \right) dt \tag{33}$$

Choosing the trial function $\theta(t) = A\cos(\omega t)$ into Eq. (33) we obtain

$$J(A,\omega) = \int_0^{T/4} \begin{pmatrix} -\frac{1}{24} l^2 \left(\mathrm{A}\,\omega \sin(\omega \mathrm{t})\right)^2 - \frac{1}{2} r^2 \left(A\cos(\omega t)\right)^2 \left(\mathrm{A}\,\omega \sin(\omega \mathrm{t})\right)^2 \\ +\frac{1}{2} r\, g \left(A\cos(\omega t)\right)^2 - \frac{1}{8} r\, g \left(A\cos(\omega t)\right)^4 + \frac{1}{144} g\, r \left(A\cos(\omega t)\right)^6 \end{pmatrix} dt \tag{34}$$

The stationary condition with respect to A reads:

$$\frac{\partial J}{\partial A} = \int_0^{T/4} \begin{pmatrix} -\frac{1}{12} l^2 \omega^2 A\, sin^2(\omega t) - 2 r^2 \omega^2 A^3 sin^2(\omega t) cos^2(\omega t) + r\, g\, A\, cos^2(\omega t) \\ -\frac{1}{2} r\, g\, A^3 cos^4(\omega t) + \frac{1}{24} r\, g\, A^5 cos^6(\omega t) \end{pmatrix} dt = 0 \tag{35}$$

or

$$\frac{\partial J}{\partial A} = \int_0^{\pi/2} \begin{pmatrix} -\frac{1}{12} l^2 A\, sin^2 t\, \omega^2 - 2 r^2 \omega^2 A^3 sin^2 t\, cos^2 t + r\, g\, A\, cos^2 t \\ -\frac{1}{2} r\, g\, A^3 cos^4 t + \frac{1}{24} r\, g\, A^5 cos^6 t \end{pmatrix} dt = 0 \tag{36}$$

Then we have;

$$\omega^2 = \frac{\int_0^{\pi/2} \left(r\, g\, A\, cos^2 t - \frac{1}{2} r\, g\, A^3 cos^4 t + \frac{1}{24} r\, g\, A^5 cos^6 t \right) dt}{\int_0^{\pi/2} \left(\frac{1}{12} l^2 A\, sin^2 t + 2 r^2 A^3 sin^2 t\, cos^2 t \right) dt} \tag{37}$$

Solving Eq. (37), according to ω, we have:

$$\omega_{VA} = \frac{1}{4} \sqrt{\frac{r\, g\, \left(192 - 72A^2 + 5A^4\right)}{6A^2 r^2 + l^2}} \tag{38}$$

Hence, the approximate solution can be readily obtained:

$$\theta(t) = A\cos\left(\frac{1}{4} \sqrt{\frac{r\, g\, \left(192 - 72A^2 + 5A^4\right)}{6A^2 r^2 + l^2}}\; t \right) \tag{39}$$

For comparison of the approximate solution, frequency obtained from solution of nonlinear equation with the energy balance method is:

$$\omega_{EBM} = \frac{\sqrt{6}}{12}\sqrt{\frac{r\,g\left(288-108A^2+7A^4\right)}{6A^2r^2+l^2}} \tag{40}$$

The numerical solution by with 4th order Runge-Kutta (Appendix A) method for nonlinear equation is:

$$\begin{aligned} &\dot{\theta} = y && \theta(0)=A \\ &\dot{y} = -\frac{r^2\theta u^2 + r\,g\,\theta\cos(\theta)}{\frac{1}{12}l^2 + r^2\theta^2} && y(0)=0 \end{aligned} \tag{41}$$

4. Result and discussion

In this section we describe the figures and tables of the comparisons. Tables 1 to 3 are the nonlinear frequency comparisons between variational approach (VA) and energy balance method (EBM) for examples 1, 2 and 3. The frequencies are very close together for different parameters of the problems.

Fig. 3 for example 1 and Fig. 7 for example 2 and figure 11 for example 3 are the displacement time history and velocity time history of the VA solution with the EBM and RKM solution. The problems are conservative and the motions of the systems are periodic and function of initial condition and amplitude. Figure 4 is the comparison of VA and EBM solution of nonlinear frequency corresponding to various parameters (a): $\Omega = 1$, $r = 5$, $g=10$ (b): $\Omega = 1.5$, $A = 2$, $g = 10$ for example 1.

For the second example we have figure 8 for the comparison of VA and EBM solution for nonlinear frequency corresponding to various parameters (i): $a = 0.1$, $b = 0.2$, $c = 0.3$, $d = 0.4$ (ii): $A = 1$, $a = 0.1$, $b = 0.2$, $c = 0.3$.

Table 1 Comparison of two approximate VA and EBM solution of nonlinear frequency corresponding to various parameters of system (example 1)

	Constant parameter				Nonlinear frequency	
No	A	Ω	r	g	VA	EBM
1	0.2	1	0.5	10	4.3497	4.3497
2	0.4	1.5	2	10	1.6809	1.6810
3	0.6	0.5	3	10	1.7251	1.7251
4	0.8	1.5	1	10	2.7569	2.7577
5	1	2	1.5	10	1.8804	1.8854
6	1.2	1	2.5	10	1.6880	1.6907
7	1.4	2	0.5	10	3.7328	3.7413
8	1.6	2.5	1	10	2.3975	2.4300
9	1.8	0.5	1.5	10	1.9872	1.9895
10	2	1	2	10	1.5789	1.5943

Table 2 Comparison of two approximate VA and EBM solution of nonlinear frequency corresponding to various parameters of system (example 2)

	Constant parameter					Nonlinear frequency	
No	A	a	b	c	d	VA	EBM
1	0.2	0.1	0.3	0.4	0.5	0.558459	0.558429
2	0.4	0.2	0.2	0.3	0.2	0.485214	0.484998
3	0.6	0.4	1	0.5	1	1.065049	1.062681
4	0.8	1.2	0.4	1	1.2	0.926177	0.918153
5	1	0.5	0.6	0.1	0.7	0.943398	0.930949
6	1.2	0.7	0.3	0.5	0.3	0.903892	0.894308
7	1.4	0.8	0.8	0.6	0.6	1.323003	1.302498
8	1.6	1	0.9	1	0.1	1.190164	1.185122
9	1.8	0.9	0.2	0.3	0.3	1.08569	1.060819

Table 3 Comparison of two approximate VA and EBM solution of nonlinear frequency corresponding to various parameters of system (example 3)

	Constant parameter				Nonlinear frequency	
No	A	r	l	g	VA	EBM
1	0.2	1	2	10	5.2800	5.2800
2	0.4	0.5	1.5	10	4.7610	4.7608
3	0.6	0.5	3	10	2.3370	2.3367
4	0.8	1.5	5	10	2.0307	2.0297
5	1	0.5	2.5	10	2.2451	2.2421
6	1.2	2	2	10	1.7886	1.7823
7	1.4	3	1	10	1.1091	1.0989
8	1.6	0.5	4	10	0.7982	0.7763
9	1.8	3	6	10	0.3156	0.2618

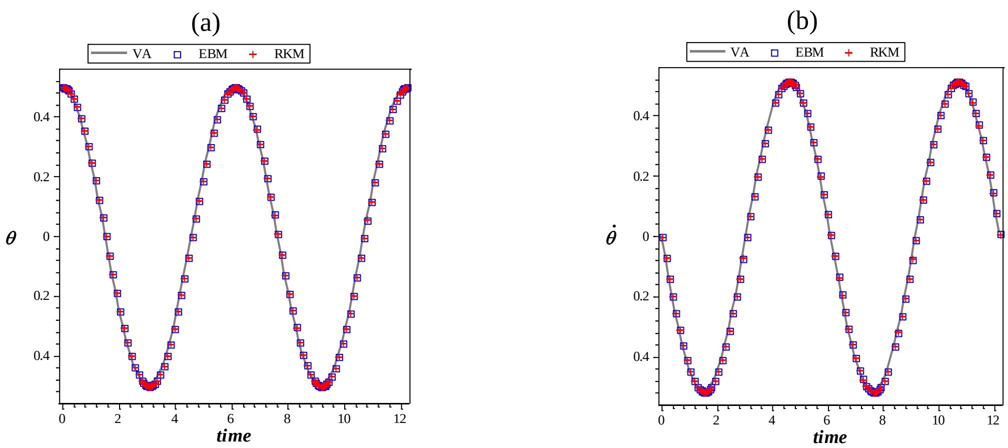

Fig. 3 Comparison of (a) displacement (b) velocity of the VA solution with the EBM and RKM solution for Ω =1, r= 5, g=10,A=0.5

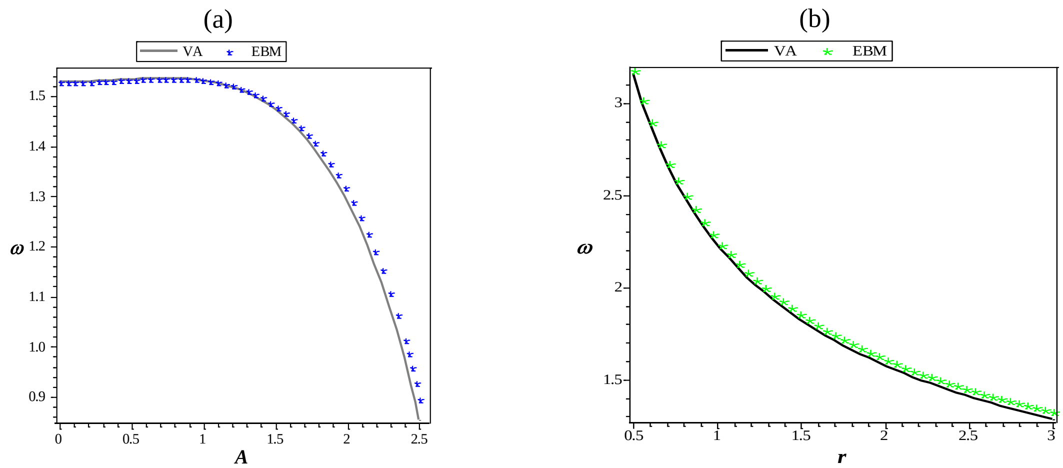

Fig. 4 Comparison between VA and EBM solution of nonlinear frequency corresponding to various parameters (a): Ω =*1*, *r*= *5*,*g*=*10* (b): Ω =1.5, *A*= 2, *g*=10

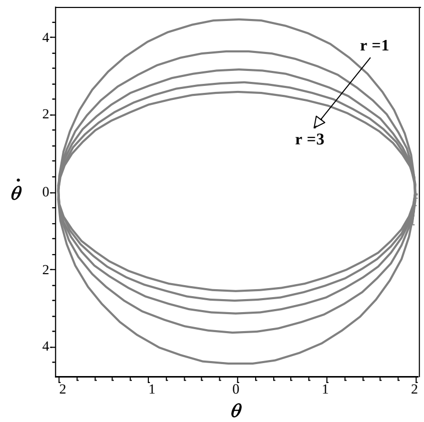

Fig. 5 Effect of parameter *r* on phase plane for Ω = 1.5,*A* = 2, *g* = 10

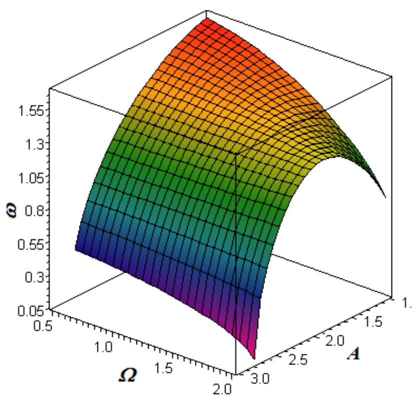

Fig.6 Sensitivity analysis of various parameter of system on nonlinear frequency

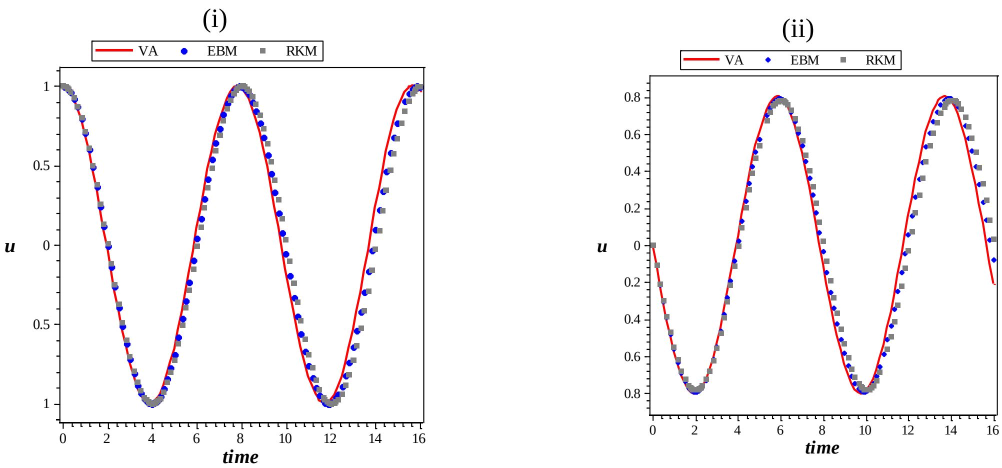

Fig. 7 Comparison of (i) displacement (ii) velocity of the VA solution with the EBM and RKM solution for A = 1, a = 0.1, b= 0.2, c = 0.3, d = 0.4,

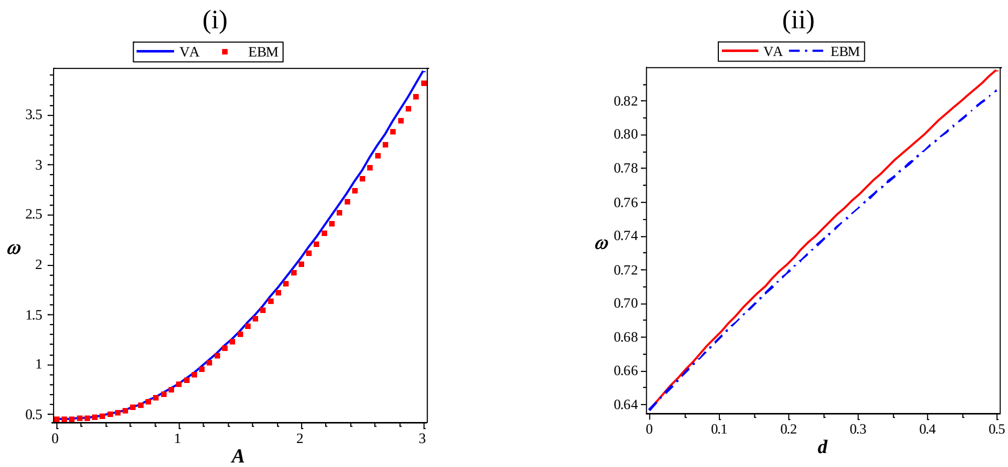

Fig. 8 Comparison between VA and EBM solution of nonlinear frequency corresponding to various parameters (i): a = 0.1, b = 0.2, c = 0.3, d = 0.4 (ii): A = 1, a = 0.1, b = 0.2, c = 0.3

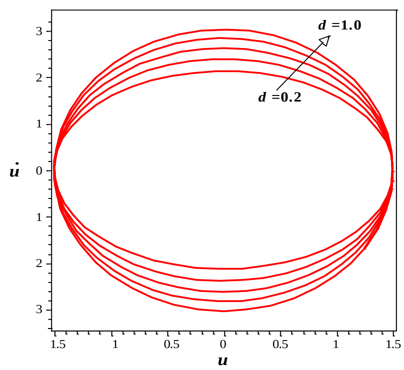

Fig. 9 Effect of parameter d on phase plane for A = 1.5, a = 0.2, b = 1, c = 0.5

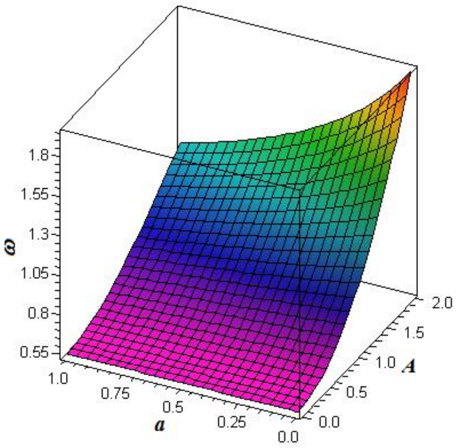

Fig. 10 Sensitivity analysis of various parameter of system on nonlinear frequency

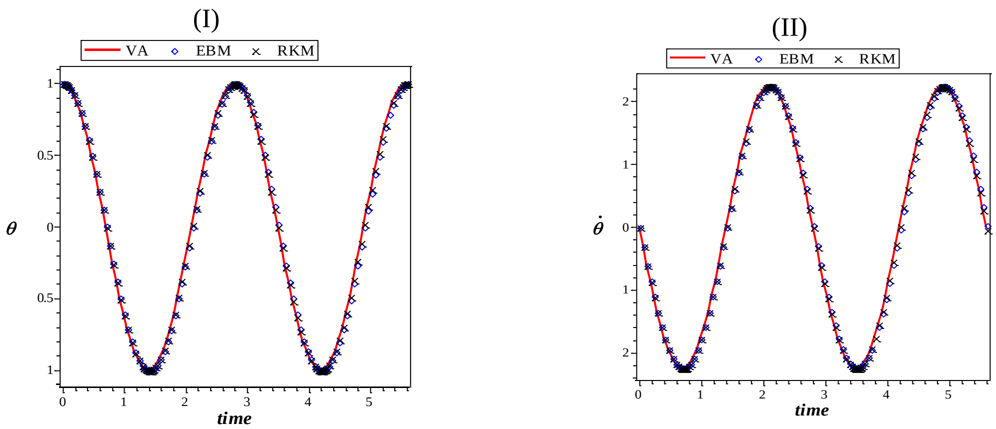

Fig. 11 Comparison of (I) displacement (II) velocity of the VA solution with the EBM and RKM solution for A = 1, l = 2.5, r = 0. 5, g = 10

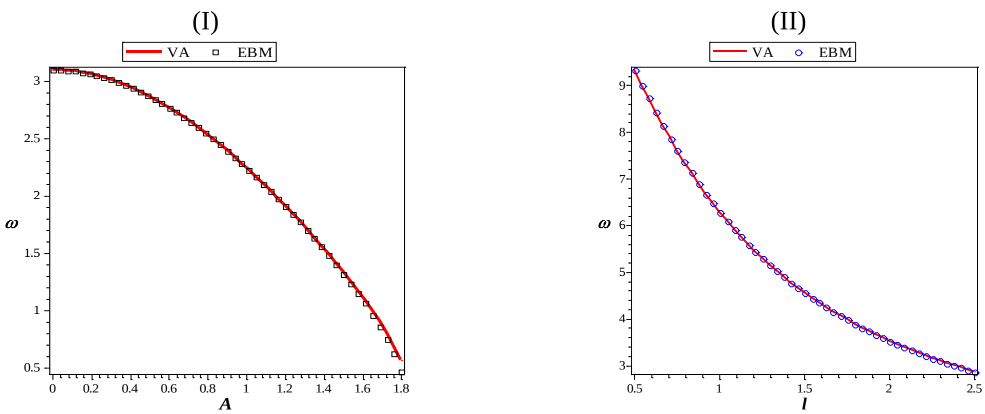

Fig. 12 Comparison between VA and EBM solution of nonlinear frequency corresponding to various parameters (I):l = 2.5, r = 0. 5, g = 10(II): A = 0.5, r = 0. 5, g = 10

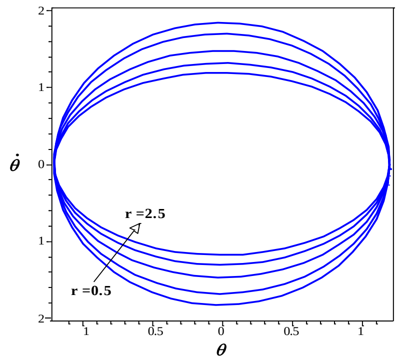

Fig. 13 Effect of parameter r on phase plane for A = 1.5, l = 2,g = 10

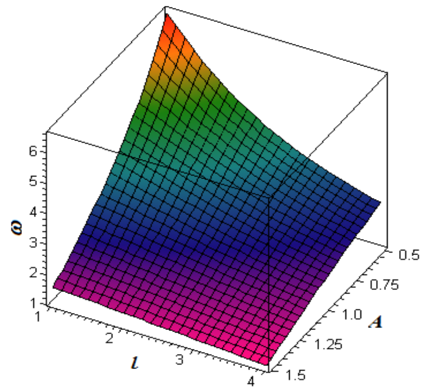

Fig. 14 Sensitivity analysis of various parameter of system on nonlinear frequency

In the third example the figure 12 is the same comparison with the following parameters: (I): l = 2.5, r = 0. 5, g = 10 (II): A = 0.5, r = 0. 5, g = 10.

Phase plans of the problems are shown in Figs. 5 for example 1, Fig. 9 for example 2 and figure 13 for example 3.

To have better understanding from the effects of important parameters on the nonlinear frequencies of the systems sensitive analyzes are done on the problems which are in figures 6,10 and 14 for examples 1 to 3 .

In this paper, it has been shown that the results of variational approach are in good agreement with energy balance method and Runge-Kutta's algorithm. The variational approach canbe easily extend to conservative nonlinear oscillations.

Appendix A: Basic idea of Runge-Kutta's Method (RKM)

For such a boundary value problem given by boundary condition, some numerical methods have been developed. Here we apply the fourth-order Runge-Kutta's algorithm to solve governing equation subject to the given boundary conditions. RK iterative formulae for the second-order differential equation are:

$$\dot{u}_{(i+1)} = \dot{u}_i + \frac{\Delta t}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$
$$u_{(i+1)} = u_i + \Delta t \left[\dot{u}_i + \frac{\Delta t}{6}\left(k_1 + k_2 + k_3\right)\right], \tag{A.1}$$

where Δt is the increment of the time and k_1, k_2, k_3 and k_4 are determined from the following formula:

$$k_1 = f\left(t_i, u_i, \dot{u}_i\right),$$
$$k_2 = f\left(t_i + \frac{\Delta t}{2}, u_i + \frac{\Delta t}{2}, \dot{u}_i + \frac{\Delta t}{2}k_1\right),$$
$$k_3 = f\left(t_i + \frac{\Delta t}{2}, u_i + \frac{\Delta t}{2}\dot{u}_i, \frac{1}{4}\Delta t^2 k_1, \dot{u}_i + \frac{\Delta t}{2}k_2\right), \tag{A.2}$$
$$K_4 = f\left(t_i + \Delta t, u_i + \Delta t\,\dot{u}_i + \frac{1}{2}\Delta t^2 k_2, \dot{u}_i + \Delta t\, K_3\right),$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative is determined from initial condition. Then, with a small time increment [Δt], the displacement function and its first-order derivative at the new position can be obtained using (A.1). This process continues to the end of time.

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