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Dynamic state estimation for identifying earthquake support motions in instrumented structures

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Abstract. The problem of identification of multi-component and (or) spatially varying earthquake support motions based on measured responses in instrumented structures is considered. The governing equations of motion are cast in the state space form and a time domain solution to the input identification problem is developed based on the Kalman and particle filtering methods. The method allows for noise in measured responses, imperfections in mathematical model for the structure, and possible nonlinear behavior of the structure. The unknown support motions are treated as hypothetical additional system states and a prior model for these motions are taken to be given in terms of white noise processes. For linear systems, the solution is developed within the Kalman filtering framework while, for nonlinear systems, the Monte Carlo simulation based particle filtering tools are employed. In the latter case, the question of controlling sampling variance based on the idea of Rao-Blackwellization is also explored. Illustrative examples include identification of multi-component and spatially varying support motions in linear/nonlinear structures.

Keywords: dynamic state estimation; particle filters; force identification; earthquake support motions

1. Introduction

Dynamic state estimation tools provide powerful framework to reconcile mathematical and experimental models in engineering dynamics (see, for example, Maybeck 1979, 1982). Thus, for a Markovian discrete time system with $n \times 1$ state vector x_k at time t_k, and a set of measurement time histories up to time t_k, denoted by D_k, the dynamic state estimation method provides a framework to determine the conditional probability density function (pdf) $p(x_k|D_k)$. In earthquake engineering problems, the equation governing system states can be derived based on the application of the finite element method and the set D_k typically could include measured structural displacements, velocities, accelerations, strains, reaction transferred, and (or) applied support motions. Prior to the availability of the measurements D_k, the unconditional pdf $p(x_k)$ can be determined by postulating a random process model for future excitations and by using random vibration principles or through Monte Carlo simulations. Once the structure comes into existence, and measurements D_k become available, it would be of interest to obtain the posterior pdf $p(x_k|D_k)$. Currently, the dynamic state estimation tools are increasingly attracting the attention of structural

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engineers given the recent advent of instrumented civil engineering structures (Ching *et al.* 2006, Yuen 2010, Nasarellah and Manohar 2011a, b). The well-known Kalman filter provides the exact solution to the problem of state estimation in linear state space models with additive Gaussian noises. For more general class of models involving nonlinear systems and (or) non-Gaussian noises, one could utilize either approximate analytical solutions (Jazwinski 1970) or Monte Carlo simulation based approaches (Doucet *et al.* 2001, Ristic *et al*. 2004, Cappé *et al.* 2005). It is important to note that the scope of the dynamic state estimation problem could be expanded to include problems of system parameter estimation and applied force identification. Some of the early applications of these tools in structural engineering can be found in the works of Yun and Shinozuka (1980) and Hoshiya and Saito (1984) who used these tools in problems of system parameter identification. More recently, several authors have explored the simulation based state estimation tools for structural system identification and model updating (Ching *et al.* 2006, Nasrellah and Manohar 2011a, b) and for reliability model updating (Ching and Beck 2007, Radhika and Manohar 2010). The present study belongs to this genre and we address the problem of identification of earthquake support motions based on a set of spatially incomplete response measurements from an existing instrumented structure.

The problem of force identification based on measured structural response is relevant in several engineering contexts such as in problems of moving load identification on bridges, impact forces on structures, forces due to imbalance in machinery, drilling applications, and wave forces on offshore structures. In structural health monitoring studies, the problem of system identification becomes central and many of the methods proposed in this context can be extended to address the problem of force identification as well. The lack of information on system condition and applied forces often results in addressing these problems simultaneously (Wang and Haldar 1994, Shi *et al.* 2000, Xu *et al.* 2012). The available methods for force identification can be broadly classified depending on three criteria: (1) time and frequency domain methods, (2) model based or measurement based or a combination of both, and (3) type of force being identified. The methods developed can belong to one or more of these classifications. A purely measurement based non-parametric identification procedure called sum of weighted acceleration technique was used by Kreitinger *et al.* (1992) in the context of linear systems where the forces are partially observed. Here each of the recorded accelerations was associated with a weight which is obtained by the method of least squares by minimizing the difference between the predicted force and the known applied force. Wang and Kreitinger (1994) extended the application of this method to nonlinear systems by formulating the identification problem in the frequency domain. Worden *et al.* (1994) proposed higher order frequency response function and time series representations for the identification of wave forces on slender cylinders. Shi *et al.* (2000) applied the Kalman filter algorithm in the frequency domain and proposed a strategy to identify the power spectral density function of the force by considering both parametric and non-parametric models for the unknown force. This method is valid for the study of linear systems. Within the context of linear systems an iterative scheme for force identification is proposed by Chen and Li (2004) where in the force identification involves a modification procedure which uses the prior information on spatial distribution of external forces acting on the structure. The next set of methods combine both model and measurement information within the Bayesian framework. As discussed above Shi *et al.* (2000) formulated the identification problem in the frequency domain and apply the Kalman filter to estimate the unknown force. Zhang *et al.* (2012) also adopt the frequency domain in their work and use the Markov Chain Monte Carlo based Gibbs' sampler to identify the parameters of the forces. Lourens *et al.* (2012) utilize the Kalman filter by augmenting the unknown force as an additional state and modeling it as a stochastic process whose covariance is determined using standard regularization parameter estimation techniques.

From the review of literature it is observed that the application of dynamic state estimation methods for force identification has been limited to studies on linear systems. Moreover, even for linear systems, the problem of identification of multicomponent and spatially varying ground motions seem to have remained unaddressed. The importance of including spatial variability in seismic ground motions in earthquake engineering has been well recognized in the existing literature (see, for example, the works of Harichandran and Vanmarcke 1986, Der Kiureghian 1995, Sugiyama *et al.* 1995, and Jankowski 2012). It is also of interest to note that methods based on conditional simulations have also been used in the existing literature to characterize earthquake ground motions based on measured ground responses (see, for example, the works of Vanmarcke and Fenton 1991, Zerva and Shinozuka 1991, Kameda and Morikawa 1994, and Jankowski *et al.*1997). In the present study we aim to develop Kalman and particle filtering based strategies for identification of earthquake induced support motions in instrumented structures by taking into account issues such as (a) transient nature of earthquake ground accelerations, (b) possible nonlinear behavior of the structure, (c) multicomponent and spatially varying support motions, and (d) imperfections in mathematical models for the structure and presence of noise in measurements. Illustrative examples include studies on a bending-torsion coupled building frame and a system with hereditary nonlinearities.

2. Problem statement

Consider an L-degree of freedom (dof) vibrating system with displacement vector *U(t)* governed by the equation

$$M\ddot{U}(t) + f\left[U(t),\dot{U}(t),t\right] = G_F F(t) + w(t); U(0) = U_0; \dot{U}(0) = \dot{U}_0 \tag{1}$$

Here a dot represents derivative with respect to time t; $U(t)$, U_0, f, and $w(t)$ are $L\times 1$ vectors; M is the $L\times L$ mass matrix; G_F is a $L\times L_F$ force co-efficient matrix; $F(t)$ is a $L_F\times 1$ vector random process representing the ground motion, with one realization of $F(t)$ corresponding to one episode of loading. The quantity $w(t)$, termed as the process noise, is a vector of Gaussian white noise processes with $\mathrm{E}\left[w(t)\right]=0$ and $\mathrm{E}\left[w(t)w^t(t+\tau)\right]=\Sigma_{w_c}\delta(\tau)$ which accounts for modeling error in arriving at Eq. (1). Here $\mathrm{E}(\bullet)$ is the mathematical expectation operator. We take that the system represented by Eq. (1) is instrumented by a N_y number of sensors and a set of measurements on system response denoted by $y_k = y(t_k)$ with $t_k, k=1,2,\cdots,N$ being the time instants at which measurements are made. The measured response could include strains, displacements, velocities, accelerations, and (or) reactions transferred to supports. We take in this study that the applied seismic support motions *F(t)* are not measured. The measured responses are taken to be related to system states through a nonlinear model given by

$$y_k = \tilde{H}\left[U(t_k),\dot{U}(t_k),t_k\right] + \nu_k; k=1,2,\cdots,N \tag{2}$$

Here $\tilde{H}$ is a $N_y\times 1$ vector; ν_k is a $N_y\times 1$ Gaussian noise vector with $\mathrm{E}\left[\nu_k\right]=0$ and $\mathrm{E}\left[\nu_k\nu_j^t\right]=\Sigma_{\nu_k}\delta_{kj}$. The noise ν_k represents the combined effects of sensor noise and uncertainties in relating y_k to the system states. We also take that ν_k is independent of the process noise $w(t)$. We use the notation $y_{1:k} = \begin{pmatrix} y_1 & y_2 & \cdots & y_k\end{pmatrix}^t$ to denote the measurement data available up to time t_k. The problem on hand consists of estimating the applied actions *F(t)* based on the postulated mathematical model for the structure [Eq. (1)] and measurement equation [Eq. (2)]. Given the presence of process and measurement noises in the above state space model, it follows that *F(t)* needs to be interpreted as a random process. The problem of identifying *F(t)* can thus be posed as determining the conditional probability density function $p_F\left[f;t_k \mid y_{1:k}\right]$.

3. Solution strategy

We pose the problem of determining $p_F\left[f;t_k \mid y_{1:k}\right]$ as a problem in Bayesian filtering and the proposed solution strategy consists of three steps:

(a) Declare $F(t)$ as a hypothetical system state and construct an augmented state vector $x(t)$ given by $x(t)=\left[U(t) \quad \dot{U}(t) \quad F(t) \quad \dot{F}(t)\right]$. A prior model for $F(t)$ is taken to be given by

$$\ddot{F}(t)=\xi_F(t);F(0)\sim p(F_0) \tag{3}$$

where $\xi_F(t)$ is taken to be a zero mean Gaussian white noise process with $\mathrm{E}\left[\xi_F(t)\xi_F(t+\tau)\right]=\sigma_{\xi_F}^2\delta(\tau)$. Note that the prior model could also include a modulating function to take into account nonstationary nature of the excitation.

(b) We combine Eqs. (1) and (3) and rewrite the governing equations in the form of an Ito's stochastic differential equation (SDE) as

$$dx(t)=a\left[x(t),t\right]dt+b\left[x(t),t\right]dB_I(t);\ x(0)=x_0 \tag{4}$$

Here $x(t)$ is a $N_x\times 1$ vector $(N_x=2L+2L_F)$, $a\left[x(t),t\right]$ is a $N_x\times 1$ drift vector, $b\left[x(t),t\right]$ is a $N_x\times m$ diffusion matrix, and $dB_I(t)$ is a $m\times 1$ vector of increments of Brownian motion processes such that $\mathrm{E}\left[dB_I(t)\right]=0$ and $\mathrm{E}\left[dB_I(t)dB_I^t(t)\right]=\Sigma_{w_c}(t)dt$. Furthermore, we discretize Eq. (4) using an explicit scheme based on application of the Ito-Taylor scheme (Kloeden and Platen 1992) and obtain a discretized map of the form

$$x_k = f_{k-1}(x_{k-1})+P_{k-1}+G_{k-1}(x_{k-1})w_{k-1};k=1,2,\cdots,N \tag{5}$$

with specified initial condition x_0. Here $x_k=x(t_k)$ is a $N_x\times 1$ state vector; $f_{k-1}(x_{k-1})$ is a $N_x\times 1$ vector of nonlinear functions of the states x_{k-1}; P_{k-1} is a $N_x\times 1$ vector corresponding to deterministic input; w_{k-1} is a $N_w\times 1$ vector of Gaussian distributed random variables with $\mathrm{E}\left[w_k\right]=0$ and $\mathrm{E}\left[w_k w_j^t\right]=\Sigma_{w_k}\delta_{kj}$ where δ_{kj} is the Kronecker delta function, and $G_{k-1}(x_{k-1})$ is a $N_x\times N_w$ matrix relating x_k and w_{k-1}. It may be noted that the deterministic input vector P_{k-1} in the above equation is included assuming that the vector of applied actions $F(t)$ may be only partially unknown. For the case when it is completely unknown, $P_{k-1}=0\forall k=1,2,\cdots,N$. We rewrite the measurement equation [Eq. (2)] in terms of the new state vector x_k as

$$y_k=H_k(x_k)+\nu_k;k=1,2,\cdots,N \tag{6}$$

(c) Eqs. (5) and (6) are now in the standard state space form which enables the application of Bayesian filtering tools (Jazwinksi 1970 and Maybeck 1979). Specifically, this consists of determining the posterior pdf-s $p(x_k \mid y_{1:k});k=1,2,\cdots,N$ based on which the desired expected values and measures of dispersion could be calculated. It must be noted that the determination of

$p\left(x_k \mid y_{1:k}\right); k=1,2,\cdots,N$ automatically leads to the determination of the estimates of the applied earthquake ground accelerations since the extended state vector x_k contains the unknown excitations as components. It is well known that for linear state space models with additive Gaussian noises, the Kalman filter provides the exact solution to the problem of state estimation. For other general class of problems involving nonlinear systems, multiplicative noises, and (or) noises with non-Gaussian distributions one could employ either approximate solutions (like extended Kalman filter) or develop numerical solutions based on Monte Carlo simulations. When simulation based methods are employed, controlling the sampling variance of the estimates obtained becomes a crucial computational issue. One of the strategies to achieve this is to explore if the state space model admits a partitioning of the form $x_k = \left(x_k^l \quad x_k^n\right)^t$, where, the superscripts l and n denoting, respectively, linear and nonlinear states, such that a part of the problem can be solved exactly using the Kalman filter and the remaining using more elaborate Monte Carlo simulations (Gordon *et al.* 1993, Schön and Gustafsson 2005, Radhika and Manohar 2012). Specifically, one here writes $p\left(x_k^l, x_{0:k}^n \mid y_{1:k}\right) = p\left(x_k^l \mid x_{0:k}^n, y_{1:k}\right) p\left(x_{0:k}^n \mid y_{1:k}\right)$ and subsequently applies the Kalman filter to obtain $p\left(x_k^l \mid x_{0:k}^n, y_{1:k}\right)$ and particle filtering to estimate $p\left(x_{0:k}^n \mid y_{1:k}\right)$. For each of the linear and nonlinear states one would get prediction and updating steps; additionally, the determination of $p\left(x_k^l \mid x_{0:k}^n, y_{1:k}\right)$ requires the treatment of $x_{0:k}^n$ as though it were a measurement. We refer the reader to the recent paper by Radhika and Manohar (2012) who have outlined the details of this procedure in which the sequential importance sampling (SIS) filtering is combined with the Kalman filter and for the details of the demonstration that the proposed procedure indeed leads to reduced Monte Carlo variance.

4. Numerical illustrations

We illustrate the procedure outlined in the preceding section through a set of four examples covering linear/nonlinear systems, single degree of freedom (sdof) or multi-dof (mdof) systems, multi-component ground motions, and spatial variations in support motions. One of the computational parameters that need to be selected pertains to the noise variance associated with augmented states corresponding to the inputs to be identified [$\sigma_{\xi_F}^2$ in Eq. (3)] and this issue is examined in detail in the first example. It turns out that this choice plays a crucial role in successful implementation of the method. The subsequent examples bring out different facets of the proposed strategy. In all the examples considered it is assumed that the structural system identification step precedes the force identification step and all measurements are derived synthetically from numerical models.

4.1 Linear sdof system subject to transient support motion

The governing equation for the total displacement of a sdof mass-spring-dash pot system under support displacement $x_b(t)$ can be written in the form

$$\ddot{u} + 2\eta\omega\left(\dot{u} - \dot{x}_b\right) + \omega^2\left(u - x_b\right) = w_1(t); \dot{u}(0) = \dot{u}_0; u(0) = u_0 \tag{7}$$

We augment this equation with an additional equation modeling the prior estimate of the unknown support motion given by

$$\ddot{x}_b = w_2(t); \dot{x}_b(0) = \dot{x}_{b0}; x_b(0) = x_{b0} \tag{8}$$

In the above equations η is the system damping, ω system natural frequency, $x_b(t)$ and $\dot{x}_b(t)$ are the support displacement and velocity, respectively, and $w_i(t); i=1,2$ are zero mean Gaussian white noise processes with $\mathrm{E}[w_i(t)w_i(t+\tau)] = \sigma_i^2\delta(\tau); i=1,2$ accounting for the modeling error. Eqs. (7) and (8) are cast as an Ito's stochastic differential equation with state vector $x(t) = \{u(t) \quad \dot{u}(t) \quad x_b(t) \quad \dot{x}_b(t)\}^t$. The SDE is discretized using the order 1.5 strong Taylor scheme (Kloeden and Platen 1992) to obtain the map

$$\begin{aligned}
x_k^1 &= x_{k-1}^1 + a_{k-1}^1\Delta + \frac{1}{2}L^0 a_{k-1}^1\Delta^2 + L^1 a_{k-1}^1\Delta Z^1 + L^2 a_{k-1}^1\Delta Z^2 \\
x_k^2 &= x_{k-1}^2 + a_{k-1}^2\Delta + \sigma_1\Delta W^1 + \frac{1}{2}L^0 a_{k-1}^2\Delta^2 + L^1 a_{k-1}^2\Delta Z^1 + L^2 a_{k-1}^2\Delta Z^2 \\
x_k^3 &= x_{k-1}^3 + x_{k-1}^4\Delta + \sigma_2\Delta Z^2 \\
x_k^4 &= x_{k-1}^4 + \sigma_2\Delta W^2
\end{aligned} \tag{9}$$

with

$$\begin{aligned}
&a_{k-1}^1 = x_{k-1}^2; a_{k-1}^2 = -2\eta\omega x_{k-1}^2 - \omega^2 x_{k-1}^1 + 2\eta\omega x_{k-1}^4 + \omega^2 x_{k-1}^3; L^0 a_{k-1}^1 = a_{k-1}^2; L^1 a_{k-1}^1 = \sigma_1; L^2 a_{k-1}^1 = 0; \\
&L^0 a_{k-1}^2 = a_{k-1}^1(-\omega^2) + a_{k-1}^2(-2\eta\omega) + a_{k-1}^3(\omega^2); L^1 a_{k-1}^2 = \sigma_1(-2\eta\omega); L^2 a_{k-1}^2 = \sigma_2(2\eta\omega)
\end{aligned} \tag{10}$$

where r_p^q is used to denote the q^{th} element of the vector $r(t)$ at the time instant $t = t_p$, Δ is the time step, and ΔW^i and $\Delta Z^i; i=1,2$ are Gaussian random variables; we refer the reader to Radhika (2012) for details on derivation of the above discrete map. Furthermore, it is assumed that the displacement of the mass is measured and the measurement equation is given by

$$y_k = x_k^1 + \nu_k; \; k = 1,2,\ldots,N \tag{11}$$

Here ν_k is the measurement noise which is taken to be a sequence of Gaussian distributed independent random variables with mean zero and standard deviation σ_{ν_k}; this noise is also taken to be independent of the process noise. It may be noted that the process equation [Eq. (9)] and measurement equation [Eq. (11)] constitute a linear state space model with additive Gaussian noises. The problem of state estimation thus can be tackled exactly using the Kalman filter. The problem of force identification lies in the determination of $p(x_k^3, x_k^4 \mid y_{1:k})$ for $k = 1,2,\cdots,N$ and this is embedded as a marginal of the filtering density $p(x_k \mid y_{1:k})$. For the purpose of illustration we take that the support motion is given by the recorded ground motion during the 1940 El Centro earthquake (PEER Ground Motion Database 2000). We synthetically generate measurement data using this excitation.

As has been already noted, the choice of the characteristics of noise $w_2(t)$ plays an important role in the development of the solution. It is important to note that the Kalman filter here provides an exact solution to the dynamic state estimation problem for any value of noise parameter σ_2. This does not automatically mean that the problem of force identification gets solved in an acceptable manner. For this to happen, an appropriate choice for σ_2 needs to be made. One option here would be to treat σ_2 itself to be an additional state and identify this parameter by performing dynamic state estimation on the extended state vector. In the present study, however, we explore a

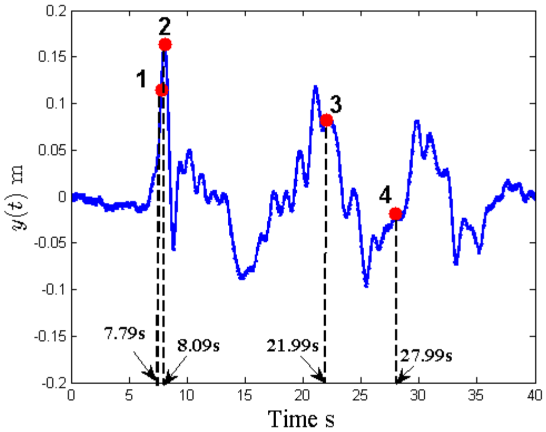


Fig. 1(a) Example in Section 4.1; Time history of the measurement with the time instants at which pdf of the estimated and measured response is compared, being marked.

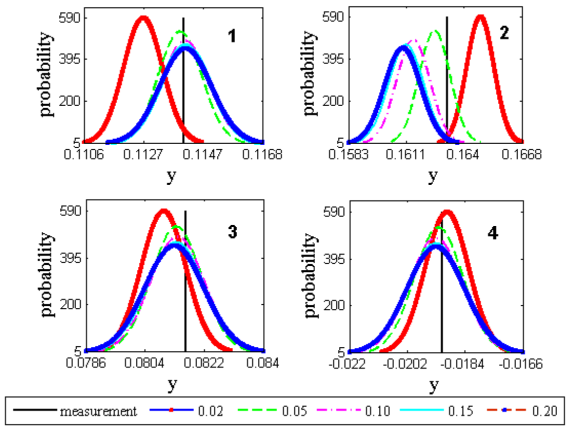
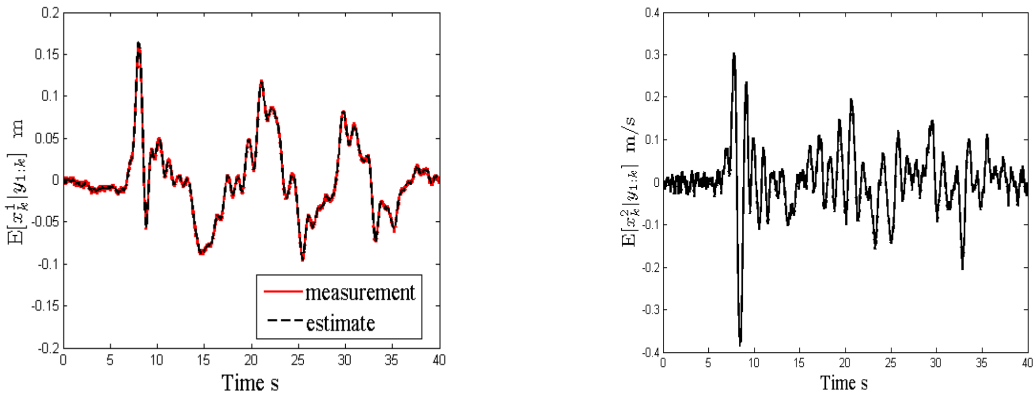


Fig. 1(b) Example in Section 4.1; comparison of the estimated and measured response for a range of values of σ_2^2; the legends correspond to different values of σ_2^2 [in $(\mathrm{m/s^2})^2$].

heuristic alternative in which we examine the results of dynamic state estimation by varying σ_2 over a range of values, with an aim to arrive at the smallest possible value of σ_2 which leads to an acceptable solution. To illustrate this we consider a system with $\omega = 2\pi$ rad/s, $\eta = 0.05$, $\sigma_1^2 = 0.01(\mathrm{m/s^2})^2$, $\Delta = 0.0100\mathrm{s}$, $T = 40\mathrm{s}$ and $\sigma_{v_k} = 0.0016\mathrm{m}$. Fig. 1(a) shows the trajectory of the measured displacement. For the purpose of illustration we select $t_k = 7.79, 8.09, 21.99, 27.99\mathrm{s}$ [these time instants have been marked in Fig. 1(a)]. Fig. 1(b) shows the pdf $\mathrm{p}\left(x_k^1 \mid y_{1:k}\right)$ for the above four time instants and for $\sigma_2^2 = 0.02, 0.05, 0.10, 0.15$ and $0.20\left(\mathrm{m/s^2}\right)^2$. Also shown in these plots, through dotted vertical lines, are the measured values y_k for the four time instants considered. It may be observed that, as σ_2 increases, $\mathrm{E}\left[x_k^1 \mid y_{1:k}\right]$ converges to a constant value which depends on the value of t_k. It may be discerned that the smallest value of σ_2 to which the converged value of $\mathrm{E}\left[x_k^1 \mid y_{1:k}\right]$ gets closest to the measured y_k is $\sigma_2^2 = 0.05\ \left(\mathrm{m/s^2}\right)^2$. In further work we thus employ $\sigma_2^2 = 0.05\ (\mathrm{m/s^2})^2$. Fig. 2 shows the plots of posterior mean of the displacement and velocity responses. The measured data on displacement is also displayed in Fig. 2(a). The results on support motion identification are shown in Fig. 3 along with details of the applied actions. To investigate the success of the method, the maximum of the Fourier amplitude spectrum of the estimated force time histories is compared with that of the applied support motions. An error of 1% (support displacement) and 0.01% (support velocity) was observed in these estimates thereby lending credence to the proposed method for force identification.

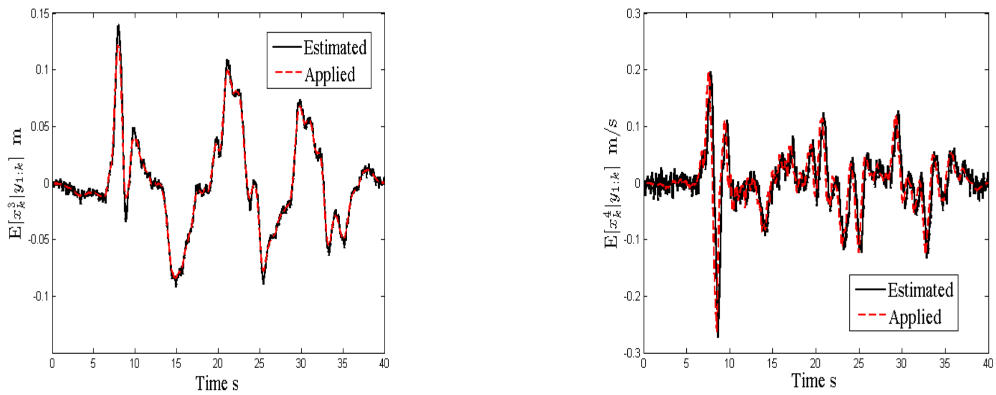
4.2 A bending-torsion coupled building model subjected to biaxial transient support motions

A five storey building frame with planar asymmetry in mass and stiffness distribution subjected to a bi-axial ground motion is considered (Fig. 4). Three of the columns are taken to be made up of steel while one of aluminum. The system is modeled as a 15 dof system (5 translations each in x and y directions and 5 rotations about the vertical axis) as shown in Fig. 4. The governing equation is obtained as



(a) (b)

Fig. 2 Example in Section 4.1; conditional mean of the estimated system responses; (a) $u(t)$; (b) $\dot{u}(t)$.



(a) (b)

Fig. 3 Example in Section 4.1; conditional mean of the estimated support motion; (a) $x_b(t)$; (b) $\dot{x}_b(t)$

$$M\ddot{U}(t)+C\dot{U}(t)+KU(t)=f(t)+w(t);\,U(0)=U_0;\,\dot{U}(0)=\dot{U}_0$$

$$U(t)=\begin{Bmatrix}x_1(t)\\y_1(t)\\\theta_1(t)\\\vdots\\x_5(t)\\y_5(t)\\\theta_5(t)\end{Bmatrix};f(t)=\begin{Bmatrix}0.5C_{11}\dot{x}_{bx}+0.5k_{11}x_{bx}\\0.5C_{22}\dot{x}_{by}+0.5k_{22}x_{by}\\0.5C_{31}\dot{x}_{bx}+0.5k_{31}x_{bx}+0.5C_{32}\dot{x}_{by}+0.5k_{32}x_{by}\\0_{12\times1}\end{Bmatrix};w(t)=\begin{Bmatrix}w_1(t)\\w_2(t)\\\vdots\\w_{15}(t)\end{Bmatrix} \tag{12}$$

Here M is a 15x15 diagonal matrix with the mass at each floor level being the co-efficient of the translational dofs$\{x_i(t),y_i(t)\}^5_{i-1}$ and the rotational inertia being the co-efficient of the rotational dofs$\{\theta_i(t)\}^5_{i-1}$.C and K are non-diagonal viscous damping and stiffness matrices, respectively. C_{ij} and K_{ij} refer to the $(i,j)^{th}$ element of the C and K matrices, respectively. $w(t)$ is a 15×1 vector of independent Gaussian white noise processes with $\mathrm{E}\left[w_i(t)\right]=0;i\in[1,15]$ and $\mathrm{E}\left[w_i(t)w_i(t+\tau)\right]=\sigma_i\delta(\tau);i\in[1,15]$. The 15 natural frequencies of the building model were obtained as 5.34, 5.53, 8.23, 15.40, 15.97, 23.52, 24.25, 24.72, 29.86, 30.86, 36.14, 39.35, 49.02 and 55.35 rad/s. x_{bx} and x_{by} are the applied support displacement in the x and y direction, respectively. Similarly, $\dot{x}_{bx}$ and $\dot{x}_{by}$ are the applied support velocities. Assuming a Gaussian white noise as a prior model for the support motions, the governing equations for the augmented states are given by

$$\ddot{x}_{bx}=w_{16}(t);\ddot{x}_{by}=w_{17}(t) \tag{13}$$

Here $\mathrm{E}\left[w_i(t)\right]=0;i=16,17$ and $\mathrm{E}\left[w_i(t)w_i(t+\tau)\right]=\sigma_i^2\delta(\tau);i=16,17$. Eqs. (12) and (13) can be recast in the Ito's SDE form and further discretized to obtain a discrete map governing the evolution of the state vector. Furthermore, it is assumed that $x_1(t),y_1(t),y_2(t),\theta_3(t),y_5(t)$ and $\theta_5(t)$ are measured, resulting in the measurement equation having the form

$$y^i_k = x^i_k + v^i_k\,;\ k = 1,2,\ldots,N;i = 1,2,5,9,10,14 \tag{14}$$

Here $v^i_k;i=1,2,5,9,10,14$ is a vector of Gaussian distributed random variables with zero mean and standard deviation 0.001 m, 0.007 m, 0.001 m, 0.0015rad/s, 0.0018 m and 0.0013 rad/s. The discretization time interval is taken to be 0.005s. Assuming classical damping a uniform damping of 5% has been assumed for all modes. The problem on hand consists of estimating $p\left(x_{bx},\dot{x}_{bx},x_{by},\dot{x}_{by}\mid y_{1:k}\right)$. As in the previous example, the governing state space model here is also linear with additive Gaussian noises and hence the problem of dynamic state estimation can be tackled exactly using the Kalman filter. By comparing the measurement with the estimated mean of the measured response the standard deviation of the excitation noises is taken to be $\sigma_{16}=0.001\mathrm{m/s^2}$ and $\sigma_{17}=0.002\mathrm{m/s^2}$. Fig. 5 shows the measurement time histories used in the identification procedure. Figs. 6 and 7 show results of the identified support motions. In Fig. 6 the mean of the estimated support displacement $x_{bx}(t)$ is shown and in Fig. 7 similar result for the mean of the estimated support velocity $\dot{x}_{by}(t)$ is shown. The estimated time histories are compared with the applied forces and an error of 0.8% and 0.1% is obtained in the peak Fourier amplitude spectrum of support displacement (x-direction) and support velocity (y-direction) respectively, implying a good match.

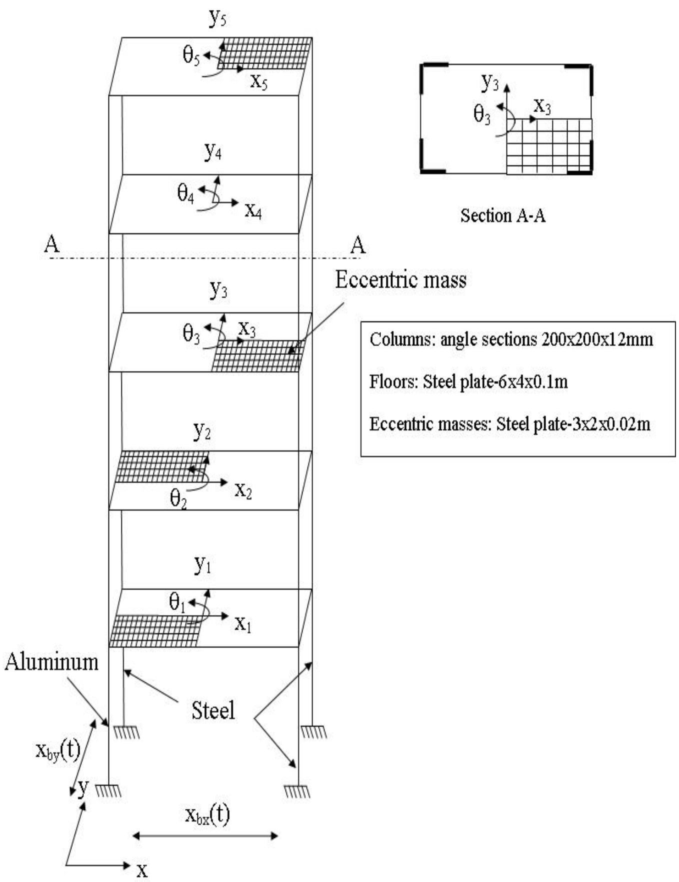


Fig. 4 Example in Section 4.2; 5-storey shear frame building model

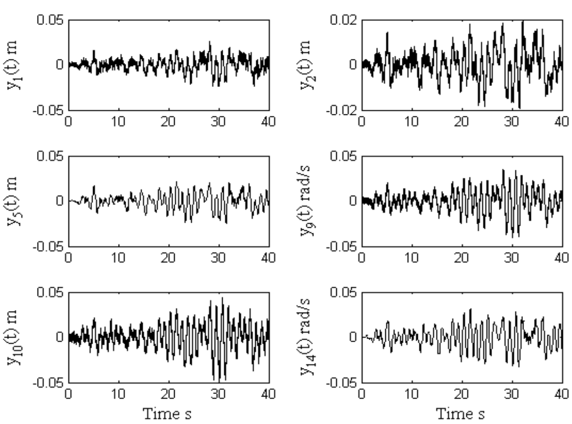


Fig. 5 Example in Section 4.2; Time history of the measurements

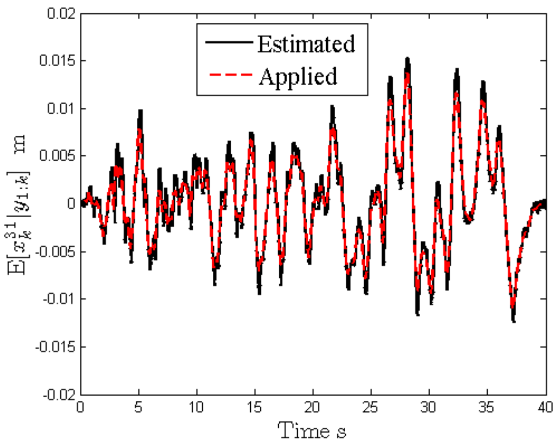


Fig. 6 Example in Section 4.2; conditional mean of the estimated support motion $x_{bx}(t)$

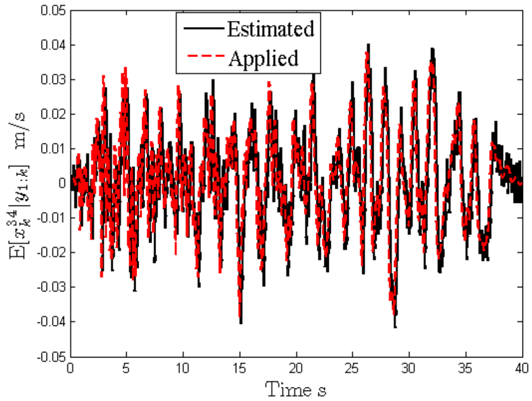


Fig. 7 Example in Section 4.2; conditional mean of the estimated support motion $\dot{x}_{by}(t)$

4.3 Euler Bernoulli beam subject to differential transient support excitations

Here we consider a simply supported Euler Bernoulli beam of span l subject to transient support motions at its left [$x_b^L(t)$] and right [$x_b^R(t)$] supports. The governing equation of motion of the beam is given by

$$
\begin{aligned}
& EIy^{iv}(s,t) + m\ddot{y}(s,t) + c\dot{y}(s,t) = 0 \\
& \text{Boundary conditions: } y(0,t) = x_b^L(t); EIy''(0,t) = 0; y(l,t) = x_b^R(t); EIy''(l,t) = 0 \\
& \text{Initial conditions: } y(s,0) = 0; \dot{y}(s,0) = 0
\end{aligned}
\tag{15}
$$

Here a prime denotes derivative with respect to spatial variable *s; EI, m, and c* are the flexural rigidity, mass of beam per unit length and the co-efficient of viscous damping, respectively. In order to make the boundary conditions time invariant, we introduce the transformation (Meirovitch

2007) $y(s,t)=z(s,t)+h_1(s)x_b^L(t)+h_2(s)x_b^R(t)$ leading to the transformed equation

$$\begin{aligned}
&EIz^{iv}(s,t)+m\ddot{z}(s,t)+c\dot{z}(s,t)=-m\left[h_1(s)\ddot{x}_b^L(t)+h_2(s)\ddot{x}_b^R(t)\right]-c\left[h_1(s)\dot{x}_b^L(t)+h_2(s)\dot{x}_b^R(t)\right]\\
&\text{Boundary conditions: } z(0,t)=0; EIz''(0,t)=0; z(l,t)=0; EIz''(l,t)=0\\
&\text{Initial conditions: } z(s,0)=-h_1(s)x_b^L(0)-h_2(s)x_b^R(0); \dot{y}(s,0)=-h_1(s)\dot{x}_b^L(0)-h_2(s)\dot{x}_b^R(0)
\end{aligned} \tag{16}$$

Here we have taken $h_1^{iv}=0$ and $h_2^{iv}=0$ and it can be shown that the choice $h_1(s)=1-\frac{s}{l}$ and $h_2(s)=\frac{s}{l}$ satisfy the boundary conditions. For the purpose of illustration, we assume that the solution is given by a 3-mode approximation $z(s,t)=\sum_{i=1}^{3}a_i(t)\sin\left(\frac{i\pi s}{l}\right)$. The governing equations for the generalized coordinates $\{a_i(t)\}_{i=1}^{3}$ is further obtained as

$$\begin{aligned}
&\ddot{a}_i(t)+2\eta_i\omega_i\dot{a}_i(t)+\omega_i^2a_i(t)=\int_0^l f(s,t)\phi_i(s)ds+w_i(t); i=1,2,\cdots,3\\
&f(s,t)=-m\left[h_1(s)\ddot{x}_b^L(t)+h_2(s)\ddot{x}_b^R(t)\right]-c\left[h_1(s)\dot{x}_b^L(t)+h_2(s)\dot{x}_b^R(t)\right] \text{ and } \omega_i=\frac{i^2\pi^2}{l^2}\sqrt{\frac{EI}{m}}
\end{aligned} \tag{17}$$

Here the white noise processes $w_i(t); i=1,2,\cdots,3$ have been added to account for the error in modeling. Furthermore, we augment the above equations with the prior model for the support motions as

$$\begin{aligned}
\ddot{x}_b^L(t)&=w_4(t)\\
\ddot{x}_b^R(t)&=w_5(t)
\end{aligned} \tag{18}$$

In Eqs. (17) and (18) we take $\mathrm{E}\left[w_i(t)\right]=0; i\in[1,5]$ and $\mathrm{E}\left[w_i(t)w_i(t+\tau)\right]=\sigma_i^2\delta(\tau); i\in[1,5]$. These equations can now be interpreted as a set of Ito's SDE-s and after discretization can be recast in the form of a linear map with additive Gaussian noise. It is assumed that the displacements at $s=\frac{l}{4},\frac{l}{3},\frac{l}{2},\frac{2l}{3},\frac{3l}{4}$ are measured and the measurement equation is given by

$$y_k=\begin{bmatrix}
\sin\left(\frac{\pi}{4}\right) & \sin\left(\frac{\pi}{2}\right) & \sin\left(\frac{3\pi}{4}\right) & 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0\\
\sin\left(\frac{\pi}{3}\right) & \sin\left(\frac{2\pi}{3}\right) & \sin(\pi) & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0\\
\sin\left(\frac{\pi}{2}\right) & \sin(\pi) & \sin\left(\frac{3\pi}{2}\right) & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0\\
\sin\left(\frac{2\pi}{3}\right) & \sin\left(\frac{4\pi}{3}\right) & \sin(2\pi) & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0\\
\sin\left(\frac{3\pi}{4}\right) & \sin\left(\frac{3\pi}{2}\right) & \sin\left(\frac{9\pi}{4}\right) & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0
\end{bmatrix}x_k+\nu_k \tag{19}$$

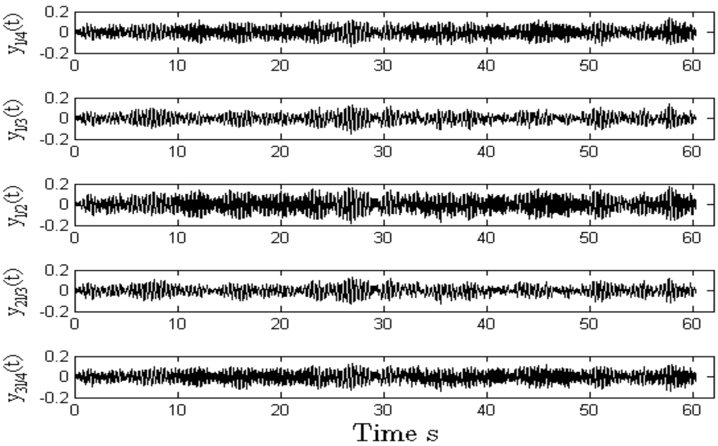


Fig. 8 Example in Section 4.3; Time history of the measurements

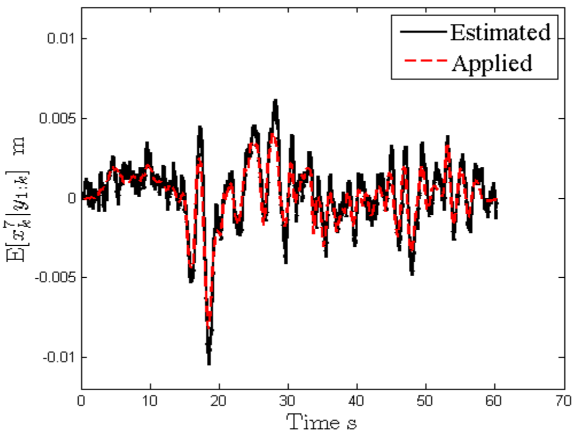


Fig. 9 Example in Section 4.3; conditional mean of the estimated left support displacement $x_b^L(t)$

Here ν_k is a 5×1 vector of independent Gaussian random variables with zero mean and standard deviations 0.0028m, 0.0030m, 0.0036m, 0.0026m and 0.0028m. In the numerical work it is assumed that the discretization step size $\Delta = 0.005$s, $l = 6$m, EI = 152.67KNm^2, $m = 6.1$kg/m, $\eta_1 = 0.02$, $\eta_2 = 0.03$ and $\eta_3 = 0.05$.

The problem of identifying the support excitations require evaluation of $p\left[x_{bx}(t_k), \dot{x}_{bx}(t_k), x_{by}(t_k), \dot{x}_{by}(t_k) \mid y_{1:k}\right]$ which is carried out using the Kalman filter. By comparing the measurement with the estimated mean of the measured response the standard deviation of the excitation processes is obtained as $\sigma_4 = \sigma_5 = 0.004\text{m/s}^2$. Fig. 8 shows the measurement time histories of the beam displacements along the span. Figs. 9 and 10 show the estimates of mean of left and right support displacements, respectively, obtained using the Kalman filter. Also shown in these plots are the time histories of the applied support motions. An error of 2.1% and 2.8% was observed in the peak Fourier amplitude spectrum of support displacements, thus validating the proposed method for force identification.

4.4 Cubic-hysteretic nonlinear system subject to transient support motion

This example serves to illustrate the input identification procedure when the instrumented structure is modeled as an inelastic system. For this purpose we consider the nonlinear system shown in Fig. 11. The system has two dof-s and consists of both memoryless and hereditary nonlinear elements. The hysteretic element is represented based on the Bouc-Wen model (Wen 1989). In this figure, $m_i\left(i=1,2\right)$ are point masses, $c_i\left(i=1,2,3\right)$ are viscous dampers, $k_i\left(i=1,2,3\right)$ are linear stiffness parameters, α is the parameter associated with cubic springs, and h is the hysteretic element. The hysteretic element h, is characterized by pre-yield stiffness k_b, and post yield stiffness λk_b, and parameters $\gamma,\bar{n},\beta$ and A which control the shape of hysteresis loops. The governing equation of the dynamical system is given by

$$\begin{aligned}
&m_1\ddot{u}_1+c_1\left(\dot{u}_1-\dot{x}_b\right)+c_2\left(\dot{u}_1-\dot{u}_2\right)+\alpha\left(u_1-x_b\right)^3+k_b\lambda\left(u_1-x_b\right)+k_bz\left(1-\lambda\right)+k_2\left(u_1-u_2\right)=w_1\left(t\right)\\
&m_2\ddot{u}_2+c_2\left(\dot{u}_2-\dot{u}_1\right)+c_3\left(\dot{u}_2-\dot{x}_b\right)+k_2\left(u_2-u_1\right)+k_3\left(u_2-x_b\right)=w_2\left(t\right)\\
&\dot{z}=-\gamma\,|\,\dot{u}_1-\dot{x}_b\,|\,z\,|\,z\,|^{\bar{n}-1}-\beta\left(\dot{u}_1-\dot{x}_b\right)|\,z\,|^{\bar{n}}+A\left(\dot{u}_1-\dot{x}_b\right)+w_3\left(t\right)\\
&\dot{u}_i\left(0\right)=0,u_i\left(0\right)=0;i=1,2;z\left(0\right)=0
\end{aligned}\tag{20}$$

Here $\dot{x}_b$ and x_b are, respectively, the velocity and displacement of the supports and $\mathrm{E}\left[w_i\left(t\right)\right]=0;\mathrm{E}\left[w_i\left(t\right)w_i\left(t+\tau\right)\right]=\sigma_i^2\delta\left(\tau\right);i=1,2,3$. The objective here is to obtain the estimates of the support displacement and velocity conditioned on the measurements on the displacement $u_1\left(t\right)$. The equation $\ddot{x}_b\left(t\right)=w_4\left(t\right)$ is taken to be the prior model for the forcing function. Eq. (20), along with this forcing model, is recast into a SDE with the state vector defined by $x\left(t\right)=\left\{u_1\left(t\right)\quad\dot{u}_1\left(t\right)\quad u_2\left(t\right)\quad\dot{u}_2\left(t\right)\quad z\left(t\right)\quad x_b\left(t\right)\quad\dot{x}_b\left(t\right)\right\}^t$. Using the order 1.5 strong Taylor scheme the SDE is discretized to obtain.

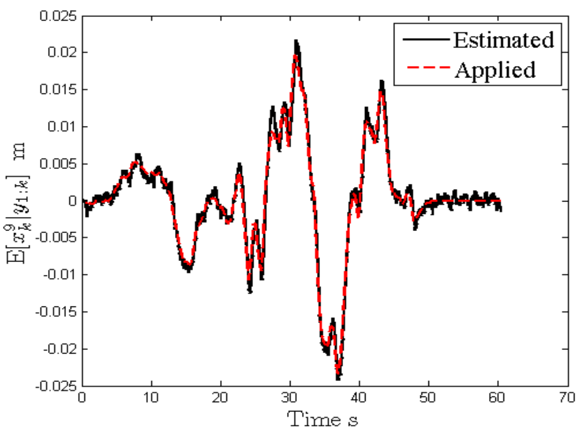


Fig. 10 Example in Section 4.3; conditional mean of the estimatedright support displacement $x_b^R\left(t\right)$

$$
\begin{aligned}
x_k^1 &= x_{k-1}^1 + a_{k-1}^1\Delta + \frac{1}{2}L^0 a_{k-1}^1\Delta^2 + L^1 a_{k-1}^1\Delta Z^1 + L^2 a_{k-1}^1\Delta Z^2 + L^3 a_{k-1}^1\Delta Z^3 + L^4 a_{k-1}^1\Delta Z^4 \\
x_k^2 &= x_{k-1}^2 + a_{k-1}^2\Delta + \frac{\sigma_1}{m_1}\Delta W^1 + \frac{1}{2}L^0 a_{k-1}^2\Delta^2 + L^1 a_{k-1}^2\Delta Z^1 + L^2 a_{k-1}^2\Delta Z^2 + L^3 a_{k-1}^2\Delta Z^3 + L^4 a_{k-1}^2\Delta Z^4 \\
x_k^3 &= x_{k-1}^3 + a_{k-1}^3\Delta + \frac{1}{2}L^0 a_{k-1}^3\Delta^2 + L^1 a_{k-1}^3\Delta Z^1 + L^2 a_{k-1}^3\Delta Z^2 + L^3 a_{k-1}^3\Delta Z^3 + L^4 a_{k-1}^3\Delta Z^4 \\
x_k^4 &= x_{k-1}^4 + a_{k-1}^4\Delta + \frac{\sigma_2}{m_2}\Delta W^2 + \frac{1}{2}L^0 a_{k-1}^4\Delta^2 + L^1 a_{k-1}^4\Delta Z^1 + L^2 a_{k-1}^4\Delta Z^2 + L^3 a_{k-1}^4\Delta Z^3 + L^4 a_{k-1}^4\Delta Z^4 \\
x_k^5 &= x_{k-1}^5 + a_{k-1}^5\Delta + \sigma_3\Delta W^3 + \frac{1}{2}L^0 a_{k-1}^5\Delta^2 + L^1 a_{k-1}^5\Delta Z^1 + L^2 a_{k-1}^5\Delta Z^2 + L^3 a_{k-1}^5\Delta Z^3 + L^4 a_{k-1}^5\Delta Z^4 \\
x_k^6 &= x_{k-1}^6 + a_{k-1}^6\Delta + \frac{1}{2}L^0 a_{k-1}^6\Delta^2 + L^1 a_{k-1}^6\Delta Z^1 + L^2 a_{k-1}^6\Delta Z^2 + L^3 a_{k-1}^6\Delta Z^3 + L^4 a_{k-1}^6\Delta Z^4 \\
x_k^7 &= x_k^7 + a_k^7\Delta + \sigma_4\Delta W^4 + \frac{1}{2}L^0 a_k^7\Delta^2 + L^1 a_k^7\Delta Z^1 + L^2 a_k^7\Delta Z^2 + L^3 a_k^7\Delta Z^3 + L^4 a_k^7\Delta Z^4
\end{aligned}
\tag{21}
$$

with

$$
\begin{aligned}
& a_{k-1}^1 = x_{k-1}^2; a_{k-1}^2 = -\left(\frac{1}{m_1}\right)\begin{bmatrix} c_1\left(x_{k-1}^2 - x_{k-1}^7\right) + c_2\left(x_{k-1}^2 - x_{k-1}^4\right) + \alpha\left(x_{k-1}^1 - x_{k-1}^6\right)^3 \\ +k_b\lambda\left(x_{k-1}^1 - x_{k-1}^6\right) + k_b x_{k-1}^5\left(1-\lambda\right) + k_2\left(x_{k-1}^1 - x_{k-1}^3\right)\end{bmatrix}; a_{k-1}^3 = x_{k-1}^4; \\
& a_{k-1}^4 = -\left(\frac{1}{m_2}\right)\left[c_3\left(x_{k-1}^4 - x_{k-1}^7\right) + c_2\left(x_{k-1}^4 - x_{k-1}^2\right) + k_3\left(x_{k-1}^3 - x_{k-1}^6\right) + k_2\left(x_{k-1}^3 - x_{k-1}^1\right)\right]; \\
& a_{k-1}^5 = -\gamma\,|\,x_{k-1}^2 - x_{k-1}^7\,|\,x_{k-1}^5\,|\,x_{k-1}^5\,|^{\bar{n}-1} - \beta\left(x_{k-1}^2 - x_{k-1}^7\right)|\,x_{k-1}^5\,|^{\bar{n}} + A\left(x_{k-1}^2 - x_{k-1}^7\right); a_{k-1}^6 = x_{k-1}^7; a_{k-1}^7 = 0; \\
& L^0 a_{k-1}^1 = a_{k-1}^2; L^1 a_{k-1}^1 = \frac{\sigma_1}{m_1}; L^2 a_{k-1}^1 = 0; L^3 a_{k-1}^1 = 0; L^4 a_{k-1}^1 = 0; \\
& L^1 a_{k-1}^2 = -\left(\frac{\sigma_1}{m_1^2}\right)\left[c_1 + c_2\right]; L^2 a_{k-1}^2 = \left(\frac{\sigma_2 c_2}{m_2 m_1}\right); L^3 a_{k-1}^2 = -\frac{\sigma_3}{m_1}\left[k_b\left(1-\lambda\right)\right]; \\
& L^4 a_{k-1}^2 = \left(\frac{\sigma_4 c_1}{m_1}\right); L^0 a_{k-1}^3 = a_{k-1}^4; L^1 a_{k-1}^3 = 0; L^2 a_{k-1}^3 = \frac{\sigma_2}{m_2}; L^3 a_{k-1}^3 = 0; L^4 a_{k-1}^3 = 0; \\
& L^0 a_{k-1}^4 = -\frac{a_{k-1}^1}{m_2}\left[-k_2\right] - \frac{a_{k-1}^2}{m_2}\left[-c_2\right] - \frac{a_{k-1}^3}{m_2}\left[k_2 + k_3\right] - \frac{a_{k-1}^4}{m_2}\left[c_2 + c_3\right] - \frac{a_{k-1}^6}{m_2}\left[-k_3\right] - \frac{a_{k-1}^7}{m_2}\left[-c_3\right]; \\
& L^1 a_{k-1}^4 = \left(\frac{\sigma_1 c_2}{m_2 m_1}\right); L^2 a_{k-1}^4 = -\left(\frac{\sigma_2}{m_2^2}\right)\left[c_2 + c_3\right]; L^3 a_{k-1}^4 = 0; L^4 a_{k-1}^4 = \left(\frac{\sigma_4 c_3}{m_2}\right); \\
& L^1 a_k^5 = \left(\frac{\sigma_1}{m_1}\right)\left[-\gamma x_{k-1}^5\,|\,x_{k-1}^5\,|^{\bar{n}-1}\operatorname{sgn}\left(x_{k-1}^2 - x_{k-1}^7\right) - \beta\,|\,x_{k-1}^5\,|^{\bar{n}} + A\right]; L^2 a_{k-1}^5 = 0; \\
& L^3 a_{k-1}^5 = \sigma_3\begin{bmatrix} -\gamma\,|\,x_{k-1}^2 - x_{k-1}^7\,||\,x_{k-1}^5\,|^{\bar{n}-1} - \gamma\left(\bar{n}-1\right)|\,x_{k-1}^2 - x_{k-1}^7\,|\,x_{k-1}^5\,|\,x_{k-1}^5\,|^{\bar{n}-2}\operatorname{sgn}\left(x_{k-1}^5\right) \\ -\beta\bar{n}\left(x_{k-1}^2 - x_{k-1}^7\right)|\,x_{k-1}^5\,|^{\bar{n}-1}\operatorname{sgn}\left(x_{k-1}^5\right)\end{bmatrix};
\end{aligned}
\tag{22}
$$

$$a_{k-1}^1 = x_{k-1}^2; a_{k-1}^2 = -\left(\frac{1}{m_1}\right)\begin{bmatrix} c_1\left(x_{k-1}^2 - x_{k-1}^7\right) + c_2\left(x_{k-1}^2 - x_{k-1}^4\right) + \alpha\left(x_{k-1}^1 - x_{k-1}^6\right)^3 \\ +k_b\lambda\left(x_{k-1}^1 - x_{k-1}^6\right) + k_b x_{k-1}^5\left(1-\lambda\right) + k_2\left(x_{k-1}^1 - x_{k-1}^3\right) \end{bmatrix}; a_{k-1}^3 = x_{k-1}^4;$$

$$a_{k-1}^4 = -\left(\frac{1}{m_2}\right)\left[c_3\left(x_{k-1}^4 - x_{k-1}^7\right) + c_2\left(x_{k-1}^4 - x_{k-1}^2\right) + k_3\left(x_{k-1}^3 - x_{k-1}^6\right) + k_2\left(x_{k-1}^3 - x_{k-1}^1\right)\right];$$

$$a_{k-1}^5 = -\gamma\,|x_{k-1}^2 - x_{k-1}^7|\,x_{k-1}^5\,|x_{k-1}^5|^{\bar{n}-1} - \beta\left(x_{k-1}^2 - x_{k-1}^7\right)|x_{k-1}^5|^{\bar{n}} + A\left(x_{k-1}^2 - x_{k-1}^7\right); a_{k-1}^6 = x_{k-1}^7; a_{k-1}^7 = 0;$$

$$L^0 a_{k-1}^1 = a_{k-1}^2; L^1 a_{k-1}^1 = \frac{\sigma_1}{m_1}; L^2 a_{k-1}^1 = 0; L^3 a_{k-1}^1 = 0; L^4 a_{k-1}^1 = 0;$$

$$L^1 a_{k-1}^2 = -\left(\frac{\sigma_1}{m_1^2}\right)\left[c_1 + c_2\right]; L^2 a_{k-1}^2 = \left(\frac{\sigma_2 c_2}{m_2 m_1}\right); L^3 a_{k-1}^2 = -\frac{\sigma_3}{m_1}\left[k_b\left(1-\lambda\right)\right];$$

$$L^4 a_{k-1}^2 = \left(\frac{\sigma_4 c_1}{m_1}\right); L^0 a_{k-1}^3 = a_{k-1}^4; L^1 a_{k-1}^3 = 0; L^2 a_{k-1}^3 = \frac{\sigma_2}{m_2}; L^3 a_{k-1}^3 = 0; L^4 a_{k-1}^3 = 0;$$

$$L^0 a_{k-1}^4 = -\frac{a_{k-1}^1}{m_2}\left[-k_2\right] - \frac{a_{k-1}^2}{m_2}\left[-c_2\right] - \frac{a_{k-1}^3}{m_2}\left[k_2 + k_3\right] - \frac{a_{k-1}^4}{m_2}\left[c_2 + c_3\right] - \frac{a_{k-1}^6}{m_2}\left[-k_3\right] - \frac{a_{k-1}^7}{m_2}\left[-c_3\right];$$

$$L^1 a_{k-1}^4 = \left(\frac{\sigma_1 c_2}{m_2 m_1}\right); L^2 a_{k-1}^4 = -\left(\frac{\sigma_2}{m_2^2}\right)\left[c_2 + c_3\right]; L^3 a_{k-1}^4 = 0; L^4 a_{k-1}^4 = \left(\frac{\sigma_4 c_3}{m_2}\right);$$

$$L^1 a_k^5 = \left(\frac{\sigma_1}{m_1}\right)\left[-\gamma x_{k-1}^5\,|x_{k-1}^5|^{\bar{n}-1}\operatorname{sgn}\left(x_{k-1}^2 - x_{k-1}^7\right) - \beta\,|x_{k-1}^5|^{\bar{n}} + A\right]; L^2 a_{k-1}^5 = 0;$$

$$L^3 a_{k-1}^5 = \sigma_3\begin{bmatrix} -\gamma\,|x_{k-1}^2 - x_{k-1}^7|\,|x_{k-1}^5|^{\bar{n}-1} - \gamma\left(\bar{n}-1\right)|x_{k-1}^2 - x_{k-1}^7|\,x_{k-1}^5\,|x_{k-1}^5|^{\bar{n}-2}\operatorname{sgn}\left(x_{k-1}^5\right) \\ -\beta\bar{n}\left(x_{k-1}^2 - x_{k-1}^7\right)|x_{k-1}^5|^{\bar{n}-1}\operatorname{sgn}\left(x_{k-1}^5\right) \end{bmatrix};$$

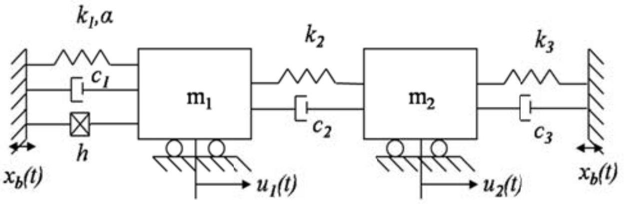


Fig. 11 Dynamical system considered in Section 4.4

The details of the derivation of these equations are provided in Radhika (2012). It is observed that the right hand side of the above equations contains terms involving Dirac's delta function. One approximate way to handle these terms in the numerical work is to represent the functions as limiting forms such as $\delta\left(x - x_0\right) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(x-x_0\right)^2}{2\sigma^2}}$. Alternatively, one needs to develop solvers that embed event location options. We are not aware of published literature that has developed this option in the context of SDE-s. In our work we adopted the former option. It was observed, however, that the response estimates were not significantly influenced by the presence of the terms corresponding to Dirac's delta function. Consequently, these terms were dropped from subsequent considerations. The resulting discrete set of equations have a structure such that the state vector can be partitioned into linear state vector $x_k^l = \left\{x_k^3 \quad x_k^4\right\}^t$ and nonlinear state vector

$x_k^n = \{x_k^1 \quad x_k^2 \quad x_k^5 \quad x_k^6 \quad x_k^7\}^t$ so that conditioned on x_k^n the state equations will have a conditionally linear Gaussian sub-structure. The measurements are assumed to be made on the displacement response $u_1(t)$ resulting in measurement equation $y_k = x_k^1 + \nu_k; k = 1, 2, \cdots, N$ where ν_k is a zero mean identical and independent Gaussian sequence of random variables (with standard deviation = 0.0029 m) which are independent of process noise terms. In the numerical work we take $m_1 = 100\text{kg}$, $m_2 = 150\,\text{kg}$, $k_b = 0.10\,\text{kN/m}$, $k_2 = 0.10\,\text{kN/m}$, $k_3 = 50\text{N/m}$, $\eta_1 = \eta_2 = 0.08$, $\alpha = 10$, $\lambda = 0.05$, $\gamma = 0.5$, $\beta = 0.5$, $A = 1$, $\overline{n} = 4$, $T = 80\text{s}$, $\Delta = 0.0100\,\text{s}$, $\sigma_1 = 0.001\text{N}$, $\sigma_2 = 0.002\,\text{N}$, $\sigma_3 = 0.001\text{N}$, and $\sigma_4 = 0.150\,\text{m/s}^2$ .The system parameter values are such that significant nonlinearity in the force-displacement relation comes into play under the applied excitations. The objective is to obtain the estimates of the conditioned support displacement $\text{E}\left[x_k^3 \mid y_{1:k}\right]$ and velocity $\text{E}\left[x_k^4 \mid y_{1:k}\right]$. Eq. (21) in conjunction with the above measurement model is amenable for solution via the SIS particle filtering method. By virtue of the partitioning of the state vector into linear and nonlinear state vectors the estimation problem can also be solved using combined Kalman-SIS filter algorithm(Radhika and Manohar 2012).In the filtering algorithm we have employed 5000 particles with $N_{thres} = 1666$. Fig. 12 shows the time history of the measured displacement. Results of the dynamic state estimation are shown in Figs. 13 and 14. Fig.13 shows the plot of conditional expectation of the displacements at the two degrees of freedom. The results of force identification are shown in Fig. 14 along with the applied support motions. The estimated support displacements resulted in an error of 0.6% (SIS) and 0.5% (Kalman-SIS) in the peak Fourier amplitude spectrum and that in estimated support velocities was observed to be 12% (SIS) and 10% (Kalman-SIS). The trends of estimated support displacement and velocity agree well with the applied actions but however are seen to be noisy especially after the strong motion phase, this being attributed to the relatively high value of $\sigma_4 = 0.150\,\text{m/s}^2$. It is to be noted that a choice of reduced value for the noise parameter will result in a poorer estimate for the support motion in the strong motion phase. This limitation could be overcome by adopting a nonstationary model for the noise so that the value of the variance of noise after the strong motion phase could be modulated. This aspect, however, requires further study. The role played by sampling fluctuations in simulation based state estimation also needs to be borne in mind in this context.

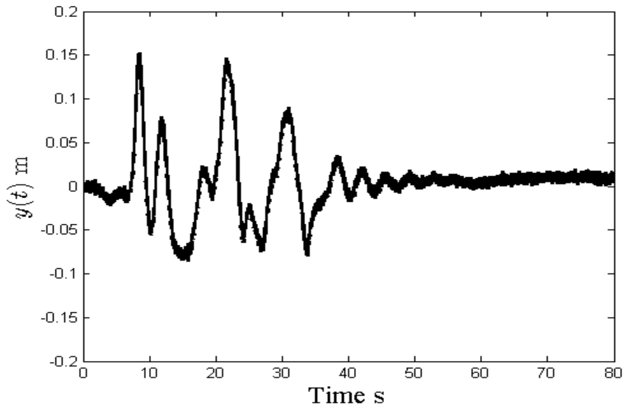
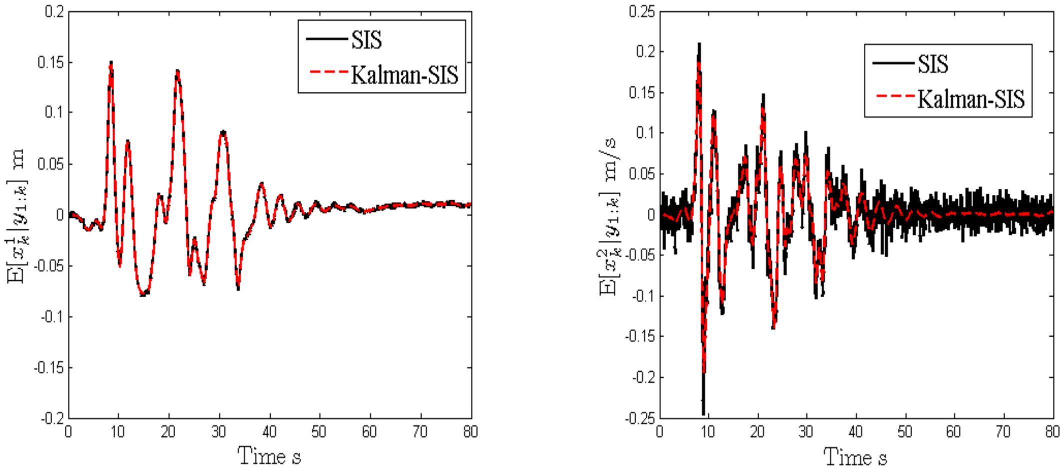
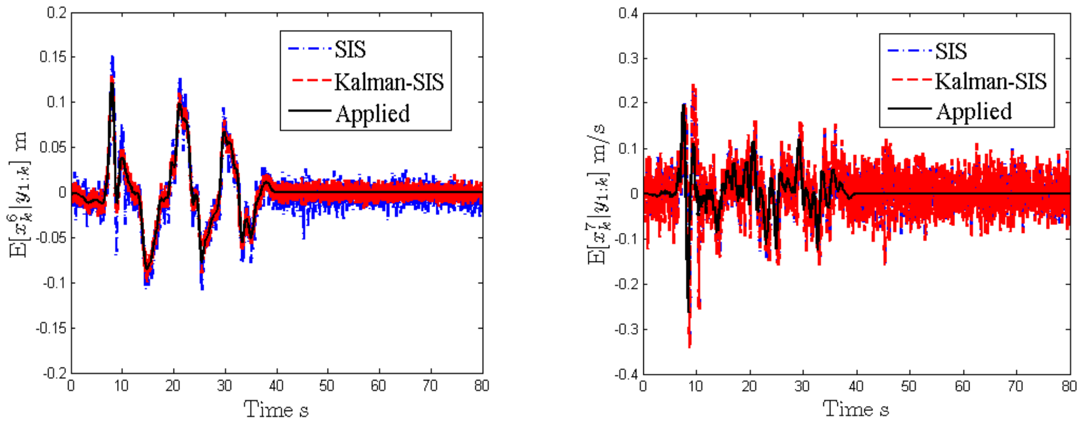


Fig. 12 Time history of measurement used in Section 4.4



(a) (b)

Fig. 13 Example in Section 4.4; estimated conditional mean of system displacement responses; (a) $u_1(t)$; (b) $u_2(t)$



(a)

Fig. 14 Example in Section 4.4; estimated conditional mean of support motions; (a) $x_b(t)$; (b) $\dot{x}_b(t)$

5. Conclusions

This paper addresses the problem of earthquake support motion identification using measured system responses within the framework of Bayesian filtering tools. The unknown forcing functions are taken as additional states modeled initially as Gaussian white noise processes and these are updated subsequently based on measured responses. The solution to the problem of dynamic state estimation for the resulting state space model is shown to contain the solution to the problem of unknown force identification. For the class of problems governed by linear state space models with additive Gaussian noises, exact solutions are obtained using the Kalman filter. For more general class of problems involving nonlinear system behavior and (or) non-Gaussian additive/multiplicative noises, particle filtering methods with provision for sampling variance reduction via importance sampling and (or) Rao-Blackwellization are shown to be effective. While for linear systems the posterior model for the excitation is Gaussian, for nonlinear systems, however, the pdf becomes non-Gaussian. Illustrative examples cover transient, multi-component and spatially varying support motions as well as linear/nonlinear system behavior and the results obtained point towards the promise of the method for further applications. It may be noted here that the identified forces using the propose method are noisy and could be considered as estimates which can be further refined for subsequent analysis. The current authors are presently exploring the application of the procedure to field examples involving large scale structures and recorded responses under realistic earthquake motions.

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