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Free vibration analysis of FG plates resting on the elastic foundation and based on the neutral surface concept using higher order shear deformation theory

Rabia Benferhat[1], Tahar Hassaine Daouadji[2], Mohamed Said Mansour[1] and Lazreg Hadji*[2]

[1]Laboratoire de Géomatériaux, Département de Génie Civil, Université de Chlef, Algérie
[2]Département de Génie Civil, Université Ibn Khaldoun, BP 78 Zaaroura, 14000 Tiaret, Algérie

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Abstract. An analytical solution based on the neutral surface concept is developed to study the free vibration behavior of simply supported functionally graded plate reposed on the elastic foundation by taking into account the effect of transverse shear deformations. No transversal shear correction factors are needed because a correct representation of the transversal shearing strain obtained by using a new refined shear deformation theory. The foundation is described by the Winkler-Pasternak model. The Young's modulus of the plate is assumed to vary continuously through the thickness according to a power law formulation, and the Poisson ratio is held constant. The equation of motion for FG rectangular plates resting on elastic foundation is obtained through Hamilton's principle. Numerical examples are provided to show the effect of foundation stiffness parameters presented for thick to thin plates and for various values of the gradient index, aspect and side to thickness ratio. It was found that the proposed theory predicts the fundamental frequencies very well with the ones available in literature.

Keywords: functionally graded material; analytical solution; free vibration analysis; neutral surface concept; elastic foundation

1. Introduction

The technique of grading ceramics along with metals initiated by the Japanese material scientist in Sendai has marked the beginning of exploring the possibility of using FGMs for various structural applications (Koizumi 1997). Since then, an effort to develop high performance heat-resistant materials using functionally gradient technology. FGMs are therefore composite materials with a microscopically inhomogeneous character. Continuous changes in their microstructure distinguish FGMs from conventional composite materials. Functionally graded materials (FGM) structures are those in which the volume fractions of two or more materials are varied continuously as a function of position along certain dimension(s) of the structure to achieve a required function. Typically, FGMs are made from a mixture of ceramic and metal. It is difficult to

*Corresponding author, Ph.D., E-mail: had_laz@yahoo.fr

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obtain an exact solution of the nonlinear equations to develop efficient mathematical models to predict the static and dynamic response of a plate. Thus far, only a few exact solutions have been investigated. However, with progress in science and technology, a need arises in engineering practice to accurately predict the nonlinear static and dynamic responses of a plate.

Plates supported by elastic foundations have been widely adopted by many researchers to model various engineering problems during the past decades. To describe the interactions of the plate and foundation as more appropriate as possible, scientists have proposed various kinds of foundation models (Kerr 1964). The simplest model for the elastic foundation is the Winkler model, which regards the foundation as a series of separated springs without coupling effects between each other, resulting in the disadvantage of discontinuous deflection on the interacted surface of the plate. This was later improved by Pasternak (Pasternak 1954) who took account of the interactions between the separated springs in the Winkler model by introducing a new dependent parameter. From then on, the Pasternak model was widely used to describe the mechanical behavior of structure-foundation interactions (Xiang 1994, Zhou 2004).

Several investigations have been presented for the analysis of FG plates and beams. Reddy (2000) use theoretical formulation, Navier's solution and finite element model for the FG plate under thermomechanical loads. Bourada *et al*. (2015) study simple shear and normal deformations theory for functionally graded beams. Mentari *et al.* (2012) use an analytical solution of the static governing equations of exponentially graded plates. Joodaky *et al.* (2013) analyze the thin skew plates made of both isotropic and functionally graded materials based on elasticity and neutral surface theory of FGMs, resting on Winkler foundation, with various combination of clamp. Senthil *et al.* (2002) provide an exact solution for three-dimensional deformations of a simply supported functionally graded rectangular plate subjected to mechanical and thermal loads on its top and/or bottom surfaces. Tounsi *et al.* (2013) use a refined trigonometric shear deformation theory for thermoelastic bending of functionally graded sandwich plates. Hien *et al.* (2014) investigate the free vibration of functionally graded material (FGM) beams on an elastic foundation and spring supports. Kamran *et al.* (2014) study the three dimensional static and dynamic analyses of two dimensional functionally graded annular sector plates. Talha *et al.* (2010) established free vibration and static analysis of functionally graded material (FGM) plates using higher order shear deformation theory with a special modification in the transverse displacement in conjunction with finite element models. Ferreira *et al.* (2005) analyzed the static deformations of a simply supported functionally graded plate modeled by a third-order shear deformation theory using the collocation multiquadric radial basis functions. Ait Yahia *et al.* (2015) analyzed the wave propagation in functionally graded plates with porosities. Ramirez *et al.* (2006) gives an approximate solution for the static analysis of three-dimensional, anisotropic, elastic plates composed of functionally graded materials by using a discrete layer theory in combination with the Ritz method in which the plate is divided into an arbitrary number of homogeneous and/or FGM layers. Thai *et al.* (2011) develop a new shear deformation plate theory for FG plates on elastic foundation which is simple to use. This theory is based on assumption that the in-plane and transverse displacements consist of bending and shear components in which the bending components do not contribute toward shear forces. Zidi *et al.* (2014) study hygro-thermo-mechanical loading for the Bending of FGM plates. Park *et al.* (2006) presented thermal postbuckling and vibration behaviors of the functionally graded (FG) plate, the nonlinear finite element equations are based on the first-order shear deformation plate theory and the von Karman nonlinear strain-displacement relationship is used to account for the large deflection of the plate.

Hadji *et al.* (2014) studied the static and free vibration of FGM beam using higher order shear deformation theory. Belabed *et al.* (2014) presented an efficient and simple higher order shear and normal deformation theory for functionally graded material (FGM) plates. Hamidi *et al.* (2015) investigated a sinusoidal plate theory with 5-unknowns and stretching effect for thermomechanical bending of functionally graded sandwich plates. Hebali *et al.* (2014) studied the static and free vibration analysis of functionally graded plates using a new quasi-3D hyperbolic shear deformation theory. Ait Amar Meziane *et al.* (2014) proposed an efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions. Mahi *et al.* (2015) studied the bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates using a new hyperbolic shear deformation theory. Bouderba *et al.* (2013) studied the thermomechanical bending response of FGM thick plates resting on Winkler-Pasternak elastic foundations. Bennoun *et al.* (2014) analyzed the vibration of functionally graded sandwich plates using a novel five variable refined plate theory.

The objective of this investigation is to present a new refined shear deformation theory to study the free vibration behavior of simply supported functionally graded plate reposed on the elastic foundation using analytical solution procedure based on the neutral surface concept. This theory does not require shear correction factors and just four unknown displacement functions are used against five or more unknown displacement functions used in the corresponding ones. The obtained results have been compared with the ones available in literature and are found to be in good agreement with them.

2. Geometric configuration and material properties

The FGM plate is regarded to be a single layer plate of uniform thickness. Here we as certain the FGM plate of length a, width *b* and total thickness h made from anisotropic material of metal and ceramics, in which the composition varies from top to bottom surface. To specify the position of neutral surface of FG plates, two different planes are considered for the measurement of *z*, namely z_{ms} and z_{ns} measured from the middle surface and the neutral surface of the plate, respectively as shown in Fig. 1.

We can write the volume fraction of ceramic (V_c) in terms of z_{ms} and z_{ns} coordinates as (Praveen 1998)

$$V_c(z) = \left(\frac{z_{ms}}{h} + \frac{1}{2}\right)^k = \left(\frac{z_{ns} + c}{h} + \frac{1}{2}\right)^k \tag{1}$$

Where *h* is the thickness of the plate and *k* denotes the power of FGM which takes values greater than or equal to zero. Also, the parameter *C* is the distance of neutral surface from the middle surface. The volume fraction of metal is expressed as

$$V_m(z) = 1 - V_c(z) \tag{2}$$

The effective Young's modulus E is expressed as (Zhang 2008)

$$E(z) = E_m V_m(z) + E_c V_c(z) \tag{3}$$

Where E_m and E_c are the Young's modulus of the metal and ceramic respectively. The position

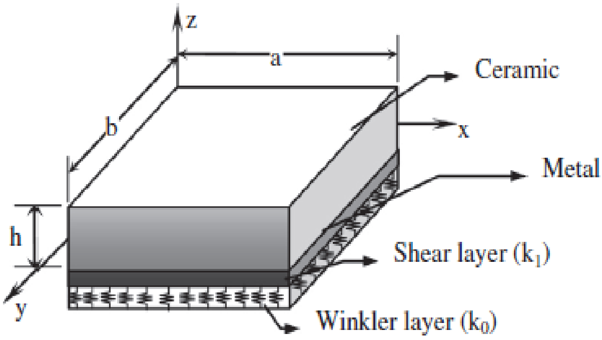

Fig. 1 Geometry and dimensions of the FGM plate resting on elastic foundation

of the neutral surface of the FG plate is determined to satisfy the first moment with respect to Young's modulus being zero as follows (Zhang 2008, Bourada *et al*. 2015, Bousahla *et al*. 2014, Al-Basyouni *et al*. 2015, Fekrar *et al*. 2014)

$$\int_{-h/2}^{h/2} E(z_{ms})(z_{ms} - C)\,dz_{ms} = 0 \tag{4}$$

Consequently, the position of neutral surface can be obtained as

$$C = \frac{\int_{-h/2}^{h/2} E(z_{ms})\,z_{ms}\,dz_{ms}}{\int_{-h/2}^{h/2} E(z_{ms})\,dz_{ms}} \tag{5}$$

It can be seen that the physical neutral surface and the geometric middle surface are the same in a homogeneous isotropic plate.

3. Displacement field and strains

In the present study, system of governing equations for FGM plate is derived by using variational approach. The origin of the material coordinates is at the neutral surface of the plate as shown in Fig 1. The in-plane displacements and the transverse displacement for the plate is assumed as

$$\begin{aligned} u(x,y,z_{ns}) &= u_0(x,y) - z_{ns}\frac{\partial w_b}{\partial x} - f(z_{ns})\frac{\partial w_s}{\partial x} \\ v(x,y,z_{ns}) &= v_0(x,y) - z_{ns}\frac{\partial w_b}{\partial y} - f(z_{ns})\frac{\partial w_s}{\partial y} \\ w(x,y,z_{ns}) &= w_b(x,y) + w_s(x,y) \end{aligned} \tag{6}$$

Where $f(z_{ns})$ represents shape functions determining the distribution of the transverse shear strains and stresses along the thickness and is given as

$$f(z_{ns}) = z_{ns} + C - \sin\left(\frac{\pi(z_{ns} + C)}{h}\right) \tag{7}$$

It should be noted that unlike the first-order shear deformation theory, this theory does not require shear correction factors. The kinematic relations can be obtained as follows

$$\begin{aligned}
\varepsilon_x &= \varepsilon_x^0 + z_{ns} k_x^b + f(z_{ns}) k_x^s \\
\varepsilon_y &= \varepsilon_y^0 + z_{ns} k_y^b + f(z_{ns}) k_y^s \\
\gamma_{xy} &= \gamma_{xy}^0 + z_{ns} k_{xy}^b + f(z_{ns}) k_{xy}^s \\
\gamma_{yz} &= g(z_{ns}) \gamma_{yz}^s \\
\gamma_{xz} &= g(z_{ns}) \gamma_{xz}^s \\
\varepsilon_z &= 0
\end{aligned} \tag{8}$$

Where

$$\begin{gathered}
\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \quad k_x^b = -\frac{\partial^2 w_b}{\partial x^2}, \quad k_x^s = -\frac{\partial^2 w_s}{\partial x^2} \\
\varepsilon_y^0 = \frac{\partial v_0}{\partial y}, \quad k_y^b = -\frac{\partial^2 w_b}{\partial y^2}, \quad k_y^s = -\frac{\partial^2 w_s}{\partial y^2} \\
\gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}, \quad k_{xy}^b = -2\frac{\partial^2 w_b}{\partial x \partial y}, \quad k_{xy}^s = -2\frac{\partial^2 w_s}{\partial x \partial y} \\
\gamma_{yz}^s = \frac{\partial w_s}{\partial y}, \quad \gamma_{xz}^s = \frac{\partial w_s}{\partial x}, \quad g(z_{ns}) = 1 - f'(z_{ns}) \text{ and } f'(z_{ns}) = \frac{df(z_{ns})}{dz_{ns}}
\end{gathered} \tag{9}$$

The constitutive relation describes how the stresses and strains are related within the plate and is expressed as

$$\begin{gathered}
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \\
\begin{Bmatrix} \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}
\end{gathered} \tag{10}$$

Where $(\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz})$ are the stress components; $(\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})$ are the strain components; Q_{ij} are the plane stress-reduced stiffnesses which can be calculated by

$$\begin{gathered}
Q_{11} = Q_{22} = \frac{E(z_{ns})}{1-\nu^2} \\
Q_{12} = \frac{\nu E(z_{ns})}{1-\nu^2} \\
Q_{44} = Q_{55} = Q_{66} = \frac{E(z_{ns})}{2(1+\nu)}
\end{gathered} \tag{11}$$

3.1 Governing equations and boundary conditions

Hamilton's principle is used herein to derive the equations of motion appropriate to the displacement field and the constitutive equations. The principle can be stated in analytical form as

$$0 = \delta \int_{t_1}^{t_2} (U + U_F - K - W)\, dt \tag{12}$$

Where U is the strain energy and K is the kinetic energy of the FG plate, U_F is the strain energy of foundation and W is the work of external forces. Employing the minimum of the total energy principle leads to a general equation of motion and boundary conditions. Taking the variation of the above equation and integrating by parts

$$\int_{t_1}^{t_2} \left[\int_V [\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx} - \rho(\ddot{u}\delta u + \ddot{v}\delta v + \ddot{w}\delta w) dv + \int_A [f_e \delta w] dA \right] dt = 0 \tag{13}$$

Where $(\cdot\cdot)$ represents the second derivative with respect to time and f_e is the density of reaction force of foundation. For the Pasternak foundation model

$$f_e = k_0 w - k_1 \nabla^2 w \tag{14}$$

Where stress and moment resultants are defined as

$$\begin{Bmatrix} N \\ M^b \\ M^s \end{Bmatrix} = \begin{bmatrix} A & 0 & B^s \\ 0 & D & D^s \\ B^s & D^s & H^s \end{bmatrix} \begin{Bmatrix} \varepsilon \\ k^b \\ k^s \end{Bmatrix}; \qquad S = A^s \gamma \tag{15}$$

In which

$$N = \{N_x, N_y, N_{xy}\}^t, \quad M^b = \{M_x^b, M_y^b, M_{xy}^b\}^t, \quad M^s = \{M_x^s, M_y^s, M_{xy}^s\}^t$$

$$\varepsilon = \{\varepsilon_x^0, \varepsilon_y^0, \varepsilon_{xy}^0\}^t, \quad k^b = \{k_x^b, k_y^b, k_{xy}^b\}^t, \quad k^s = \{k_x^s, k_y^s, k_{xy}^s\}^t$$

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}$$

$$B^s = \begin{bmatrix} B_{11}^s & B_{12}^s & 0 \\ B_{12}^s & B_{22}^s & 0 \\ 0 & 0 & B_{66}^s \end{bmatrix}, \quad D^s = \begin{bmatrix} D_{11}^s & D_{12}^s & 0 \\ D_{12}^s & D_{22}^s & 0 \\ 0 & 0 & D_{66}^s \end{bmatrix}, \quad H^s = \begin{bmatrix} H_{11}^s & H_{12}^s & 0 \\ H_{12}^s & H_{22}^s & 0 \\ 0 & 0 & H_{66}^s \end{bmatrix} \tag{16}$$

$$S = \{S_{xz}^z, S_{yz}^s\}^t, \quad \gamma = \{\gamma_{xz}, \gamma_{yz}\}^t, \quad A^s = \begin{bmatrix} A_{44}^s & 0 \\ 0 & A_{55}^s \end{bmatrix}$$

And stiffness components and inertias are given as

$$\{A_{ij}, C_{ij}, D_{ij}, E_{ij}, G_{ij}\} = \int_{-h/2-c}^{h/2-c} \{1, f(z_{ns}), z_{ns}^2, z_{ns} f(z_{ns}), [f(z_{ns})]^2\} Q_{ij}\, dz_{ns} \tag{17}$$

$$(i, j = 1,2,6)$$

$$I_1, I_2, I_3, I_4, I_5, I_6 = \int_{-h/2-c}^{h/2-c} \rho(1, z_{ns}, z_{ns}^2, f(z_{ns}), z_{ns} f(z_{ns}), [f(z_{ns})]^2)\, dz_{ns} \tag{18}$$

For FG plates, the equilibrium equations take the forms

$$\begin{aligned}
&A_{11} d_{11} u_0 + A_{66} D_{22} u_0 + (A_{12} + A_{66}) d_{12} v_0 - (B_{12}^s + 2B_{66}^s) d_{122} w_s - B_{11}^s d_{111} w_s = I_0 \ddot{u} \\
&A_{22} d_{22} v_0 + A_{66} d_{11} v_0 + (A_{12} + A_{66}) d_{12} u_0 - (B_{12}^s + 2B_{66}^s) d_{112} w_s - B_{22}^s d_{222} w_s = I \ddot{v} \\
&\quad - D_{11} d_{1111} w_b - 2(D_{12} + 2D_{66}) d_{1122} w_b - D_{22} d_{2222} w_b - D_{11}^s d_{1111} w_s \\
&\quad - 2(D_{12}^s + 2D_{66}^s) d_{1122} w_s - D_{22}^s d_{2222} w_s = I_0 (\ddot{w}_b + \ddot{w}_s) - I_2 \nabla^2 \ddot{w}_b \\
&B_{11}^s d_{111} u_0 + (B_{12}^s + 2B_{66}^s) d_{122} u_0 + (B_{12}^s + 2B_{66}^s) d_{112} v_0 + B_{22}^s d_{222} v_0 - D_{11}^s d_{1111} w_b \\
&- 2(D_{12}^s + 2D_{66}^s) d_{1122} w_b - D_{22}^s d_{2222} w_b - H_{11}^s d_{1111} w_s - 2(H_{12}^s + 2H_{66}^s) d_{1122} w_s \\
&- H_{22}^s d_{2222} w_s + A_{55}^s d_{11} w_s + A_{44}^s d_{22} w_s = I_0 (\ddot{w}_b + \ddot{w}_s) - \frac{I_2}{84} \nabla^2 \ddot{w}_b
\end{aligned} \tag{19}$$

where d_{ij}, d_{ijl}, and d_{ijlm} are the following differential operators

$$d_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \quad d_{ijl} = \frac{\partial^3}{\partial x_i \partial x_j \partial x_l}, \quad d_{ijlm} = \frac{\partial^4}{\partial x_i \partial x_j \partial x_l \partial x_m}, \quad (i, j, l, m = 1,2) \tag{20}$$

The following representation for the displacement quantities of the shear deformation theories is appropriate in the case of the free vibration problem

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_b \\ w_s \end{Bmatrix} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn} \cos(\lambda x) \sin(\mu y) e^{i\omega t} \\ V_{mn} \sin(\lambda x) \cos(\mu y) e^{i\omega t} \\ W_{bmn} \sin(\lambda x) \sin(\mu y) e^{i\omega t} \\ W_{smn} \sin(\lambda x) \sin(\mu y) e^{i\omega t} \end{Bmatrix} \tag{21}$$

Where $\lambda = m\pi / a$, $\mu = n\pi / b$ and U_{mn}, V_{mn}, W_{bmn}, W_{smn} being arbitrary parameters and ω denotes the eigenfrequency associated with (m,n)th eigenmode.

We can get the below eigenvalue equations for any fixed value of *m* and *n*, for free vibration problem

$$([K] - \omega^2 [M]\{\Delta\} = \{0\} \tag{22}$$

Where [K] and [M], stiffness and mass matrices, respectively, and represented as

$$[K]=\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{12} & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} \tag{23}$$

$$[M]=\begin{bmatrix} m_{11} & 0 & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix} \tag{24}$$

In which

$$\begin{gathered}
a_{11} = A_{11}\lambda^2 + A_{66}\mu^2 \\
a_{12} = \lambda\,\mu\left(A_{12} + A_{66}\right) \\
a_{14} = -\lambda\,[B_{11}^s\lambda^2 + (B_{12}^s + 2B_{66}^s)\,\mu^2] \\
a_{22} = A_{66}\lambda^2 + A_{22}\mu^2 \\
a_{24} = -\mu[(B_{12}^s + 2B_{66}^s)\,\lambda^2 + B_{22}^s\mu^2] \\
a_{33} = D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2\mu^2 + D_{22}\mu^4 + k_0 + k_1(\lambda^2 + \mu^2) \\
a_{34} = D_{11}^s\lambda^4 + 2(D_{12}^s + 2D_{66}^s)\lambda^2\,\mu^2 + D_{22}^s\,\mu^4 + k_0 + k_1(\lambda^2 + \mu^2) \\
a_{44} = H_{11}^s\lambda^4 + 2(H_{11}^s + 2H_{66}^s)\lambda^2\mu^2 + H_{22}^s\mu^4 - A_{55}^s\lambda^2 - A_{44}^s\mu^2 + k_0 + k_1(\lambda^2 + \mu^2) \\
m_{11} = m_{22} = -I_0,\, m_{13} = \lambda I_1,\, m_{14} = \lambda J_1,\; m_{23} = \mu I_1 \\
m_{24} = \mu J_1,\; m_{33} = -\left(I_0 + I_2\left(\lambda^2 + \mu^2\right)\right), \\
m_{34} = -\left(I_0 + J_2\left(\lambda^2 + \mu^2\right)\right), m_{44} = -\left(I_0 + K_2\left(\lambda^2 + \mu^2\right)\right), \\
\text{and } \lambda = m\pi / a,\; \mu = n\pi / b
\end{gathered} \tag{25}$$

The natural frequencies of FG plate can be found from the nontrivial solution of Eq. (22).

4. Numerical results and discussion

In this section, various numerical examples are presented and discussed to verify the accuracy of present theory in predicting the frequency of simply supported FG plates based on the neutral surface concept is taken up for investigation. For numerical results, an Al/Al_2O_3 or $Al/\ ZrO_2$ plate composed of aluminum (as metal) and alumina or Zirconia (as ceramic) is considered. The material properties assumed in the present analysis are as follows:

Ceramic

- (P_C : Alumina, Al_2O_3): $E_c = 380$ GPa; $\rho_c = 3800 kg/m^3$

- (P_C : Zirconia (ZrO_2): $E_c = 200$ GPa; $\rho_c = 5700 kg/m^3$

Metal (P_M : Aluminium, Al): $E_m = 70$ GPa; $\rho_m = 2702 kg/m^3$

Poisson's ratio is 0.3 for both alumina and aluminum. And their properties change through the thickness of the plate according to power-law. The bottom surfaces of the FG plate are aluminum rich, whereas the top surfaces of the FG plate are alumina or Zirconia rich.

For verification purpose, the obtained results are compared with Hosseini-Hashemi *et al.* (2010) based on a exact closed form Levy-type solution, Zhou *et al.* (2002) were based on a three dimensional Ritz method, Matsunaga (2008) based on the higher order shear deformation theories, three dimensional exact solution of Leissa (1973), Liu *et al.* (1999) were based on a differential quadrature element method and others available in literature.

In all examples, no transversal shear correction factors are used because a correct representation of the transversal shearing strain is given. For convenience, the following results are presented in graphical and tabular form.

To illustrate the accuracy of present theory for FG SSSS square plates made of Al/Al_2O_3 and Al/ZrO_2 for wide range of power law index *k* and thickness ratio *h/a*, the variations of Non-dimensional natural frequencies and the fundamental frequency are illustrated in the following examples.

Table 1 shows the comparison of fundamental frequency parameter ($\bar{w} = \omega h \sqrt{\rho_c / E_c}$) for SSSS Al/Al_2O_3 square plates with three values of thickness to length ratio (*h*/*a*=0.05, 0.1 and 0.2). It can be seen that the proposed refined theory using analytical solution based on the neutral surface concept and the others theories give identical results for all values of power law index *k*.

The capability of the present solution is also tested for two types of materials, plates made of

Table 1 Comparison study of fundamental frequency parameter $\bar{w} = \omega h \sqrt{\rho_c / E_c}$ for SSSS Al/Al_2O_3 square plates (a/b=1)

Thickness to length ratio *h*/*a*	Method	Gradient index *k*			
		0	1	4	10
0.05	Hosseini (2010)	0.01480	0.01150	0.01013	0.00963
	Matsunaga (2008)	-	-	-	-
	Zhao (2009)	0.01464	0.01118	0.00970	0.00931
	Present	0.01479	0.00997	0.00883	0.00810
0.1	Hosseini (2010)	0.05769	0.04454	0.03825	0.03627
	Matsunaga (2008)	0.05777	0.04427	0.03811	0.03642
	Zhao (2009)	0.05673	0.04346	0.03757	0.03591
	Present	0.05769	0.03913	0.03443	0.03150
0.2	Hosseini (2010)	0.2112	0.1650	0.1371	0.1304
	Matsunaga (2008)	0.2121	0.1640	0.1383	0.1306
	Zhao (2009)	0.2055	0.1587	0.1356	0.1284
	Present	0.2112	0.1460	0.1255	0.1142

Al/Al_2O_3 and Al/ZrO_2 for wide range of power law index k in table 2. Close correlation is achieved.

Table 3 examines the effect of Thickness to length ratio h/a on the first eight Non-dimensional natural frequencies ($\bar{w} = \omega a^2 \sqrt{\rho h / D}$) for simply supported isotropic square plate. As can be seen from the table that, not only for thin plates but also thick plates, the natural frequencies are predicted as accurately by the present method.

Tables 4 and 5 shows the comparison of fundamental frequency $\bar{w} = \omega a^2 \sqrt{\rho h / D}$ of FG rectangular plates on elastic foundation with those reported by Akhavan *et al*. (2009), Hassen Ait Atmane *et al.* (2011), Matsunaga (2008) and Thai *et al*. (2012) with different values of the

Table 2 Comparison study of fundamental frequency parameter $\bar{w} = \omega a^2 \sqrt{\rho_c / E_c} / h$ for SSSS square plates (a/b=1) when h/a=0.1

FGMs	Method	Gradient index k					
		0	1	2	5	8	10
Al/Al_2O_3	Hosseini (2010)	5.7693	4.4545	4.0063	3.7837	3.6830	3.6277
	Zhao (2009)	5.6763	4.3474	3.9474	3.7218	3.6410	3.5923
	Present	5.7696	3.9138	3.7034	3.3635	3.2093	3.1500
Al/ZrO_2	Hosseini (2010)	5.7693	5.2532	5.3084	5.2940	5.2312	5.1893
	Zhao (2009)	5.6763	4.8713	4.6977	4.5549	4.4741	4.4323
	Present	5.7696	5.0800	5.1148	5.1381	5.1156	5.1000

Table 3 Comparison study of Non-dimensional natural frequencies $\bar{w} = \omega a^2 \sqrt{\rho h / D}$ for simply supported isotropic square plate

Thickness to length ratio h/a	Method	Mode							
		1,1	1,2	2,1	2,2	3,1	1,3	3,2	2,3
0.001	Leissa (1973)	19.7392	49.348	49.348	78.9568	98.696	98.696	128.3021	128.3021
	Zhou (2002)	19.7115	49.347	49.347	78.9528	98.6911	98.6911	128.3048	128.3048
	Akavci (2014)	19.7391	49.3476	49.3476	78.9557	98.6943	98.6943	128.3020	128.3020
	Present	19.7391	49.3475	49.3475	78.9556	98.6942	98.6942	128.3018	128.3018
0.01	Liu (1999)	19.7319	49.3027	49.3027	78.8410	98.5150	98.5150	127.9993	127.9993
	Nagino (2008)	19.732	49.305	49.305	78.846	98.525	98.525	128.01	128.01
	Akavci (2014)	19.7322	49.3045	49.3045	78.8456	98.5223	98.5223	128.012	128.012
	Present	19.7320	49.3032	49.3032	78.8422	98.5171	98.5171	128.0027	128.0027
0.1	Liu (1999)	19.0584	45.4478	45.4478	69.7167	84.9264	84.9264	106.5154	106.5154
	Hosseini (2011)	19.0653	45.4869	45.4869	69.8093	85.0646	85.0646	106.7350	106.7350
	Akavci (2014)	19.0850	45.5957	45.5957	70.0595	85.4315	85.4315	107.3040	107.3040
	Present	19.0660	45.4917	45.4917	69.8212	85.0829	85.0829	106.7652	106.7652
0.2	Shufrin (2005)	17.4524	38.1884	38.1884	55.2539	65.3130	65.3130	78.9864	78.9864
	Hosseini (2011)	17.4523	38.1883	38.1883	55.2543	65.3135	65.3135	78.9865	78.9865
	Akavci (2014)	17.5149	38.4722	38.4722	55.8358	66.1207	66.1207	80.1637	80.1637
	Present	17.4553	38.2052	38.2052	55.2943	65.3731	65.3731	79.0812	79.0812

thickness to length ratios and foundation stiffness parameters. It can be seen that the results are in excellent agreement with each other.

Fundamental frequencies $\overline{w} = \omega b^2 \sqrt{SH/A}/\pi^2$ of the FG square plate (a/b=1) with simply-supported boundary conditions for h/a=0.01,0.1 and 0.2 are listed in Table 6 for different values of foundation stiffness parameters are computed and compared with other published solutions. It can be seen from the table that a good agreement is achieved between the results of present theory and those of other theory.

Figs. 2 and 3 contains the plots of non-dimensional fundamental frequency $\overline{w} = \omega a^2 / h\sqrt{\rho_m / E_m}$ of Al/Al_2O_3 functionally graded square plates with respect to power law index k (k=0 to 10)

Table 4 Comparison study of the fundamental frequency parameter $\overline{w} = \omega a^2 \sqrt{\rho h / D}$ of isotropic square plate

Thickness to length ratio h/a	K_0	K_1	Method		
			Akhavan (2009)	Hassen (2010)	Present
0.001	0	0	19.7391	19.7392	19.7320
	10^2	10	26.2112	26.2112	26.2048
	10^3	10^2	57.9961	57.9962	57.9894
0.1	0	0	19.0840	19.0658	19.0660
	10^2	10	25.6368	25.6236	25.5989
	10^3	10^2	57.3969	57.3923	57.2775
0.2	0	0	17.5055	17.4531	17.4553
	10^2	10	24.3074	24.2728	24.1068
	10^3	10^2	56.0359	56.0311	56.0260

Table 5 Comparison study of Non-dimensional natural frequencies $\overline{w} = \omega a^2 \sqrt{\rho h / D}$ for simply supported isotropic square plate resting on elastic foundation (h/b=0.2)

K_0	K_1	$\hat{\omega}_{11}$			$\hat{\omega}_{12}$			$\hat{\omega}_{13}$		
		Matsunaga (2000)	Akavci (2014)	Present	Matsunaga (2000)	Akavci (2014)	Present	Matsunaga (2000)	Akavci (2014)	Present
0	0	17.5260	17.5149	17.45533	38.4827	38.4722	38.2052	65.9961	66.1207	65.3731
10		17.7847	17.7859	17.7196	38.5929	38.5929	38.3203	66.0569	66.1899	65.4378
10^2		19.9528	20.0603	19.9413	39.5669	39.6620	39.3417	66.5995	66.8087	66.0178
10^3		34.3395	35.5261	35.1278	47.8667	47.0757	48.3829	71.5577	72.6997	71.5586
10^4		45.5260	45.5260	45.5260	71.9829	71.98297	1.98299	97.4964	101.7990	101.79922
10^5		45.5260	45.5260	45.5260	71.9829	71.9830	71.9829	101.7992	101.7990	101.7992
0	10	22.0429	22.2607	22.0950	43.4816	44.0294	43.5262	71.4914	72.6178	71.4814
10		22.2453	22.4745	22.3043	43.5747	44.1347	43.6274	71.5423	72.6806	71.5406
10^2		23.9830	24.3133	24.1068	44.3994	45.0711	44.5271	71.9964	73.2430	72.0713
10^3		36.6276	38.0839	37.6468	51.6029	53.5296	52.6856	76.1848	78.6389	77.1762
10^4		45.5260	45.5260	45.5260	71.9829	71.9829	71.9829	99.0187	101.7990	101.7992
10^5		45.5260	45.5260	45.5260	71.9829	71.9829	71.9829	101.7992	101.7990	101.7992

Table 6 Comparison study of fundamental frequency parameter $\overline{w} = \omega b^2 \sqrt{Sh/A}/\pi^2$ * for homogeneous SSSS square plates (a/b=1)

Thickness to length ratio h/a	Method	Fundamental frequency parameter			
Foundation stiffness parameters (K_0, K_1)		(100,0)	(500,0)	(100,10)	(500,10)
0.01	Hosseini (2010)	2.2413	3.0215	2.6551	3.3400
	Xiang (1994)	2.2413	3.0215	2.6551	3.3400
	Zhou (2004)	2.2413	3.0214	2.6551	3.3398
	Present	2.2413	3.0214	2.6551	3.3399
Foundation stiffness parameters (K_0, K_1)		(200,0)	(1000,0)	(200,10)	(1000,10)
0.1	Hosseini (2010)	2.3989	3.7212	2.7842	3.9805
	Xiang (1994)	2.3989	3.7212	2.7842	3.9805
	Zhou (2004)	2.3951	3.7008	2.7756	3.9566
	Present	2.3971	3.7153	2.7811	3.9738
Foundation stiffness parameters (K_0, K_1)		(0,10)	(10,10)	(100,10)	(1000,10)
0.2	Hosseini (2010)	2.2505	2.2722	2.4590	3.8567
	Xiang (1994)	2.2505	2.2722	2.4591	3.8567
	Zhou (2004)	2.2334	2.2539	2.4300	3.7111
	Present	2.2386	2.2599	2.4425	3.8144

* $S = (k(8+3k+k^2)\rho_m+3(2+k+k^2)\rho_c)/(1+k)/(2+k)/(3+k)$;

$A =$ $1/12(h^3/(1-\nu^2)(k(8+3k+k^2)E_m+3(2+k+k^2)E_c)/(1+k)/(2+k)/(3+k)$

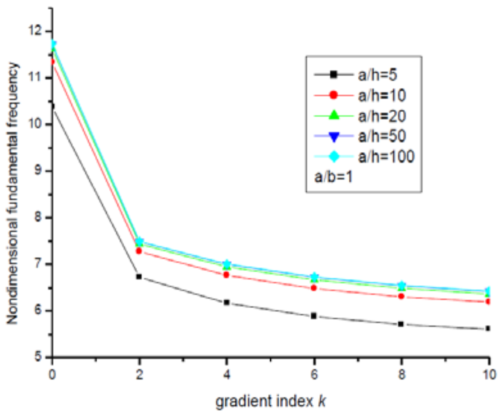

Fig. 2 Non-dimensional fundamental frequency $\overline{w} = \omega a^2/h\sqrt{\rho_m/E_m}$ of Al/Al_2O_3 across power law index k

without the elastic foundation (K_0=K_1=0). It is clear that the increase of the power law index k causes a decrease of the non-dimensional fundamental frequency. This last increase when the aspect and side-to-thickness ratio increase.

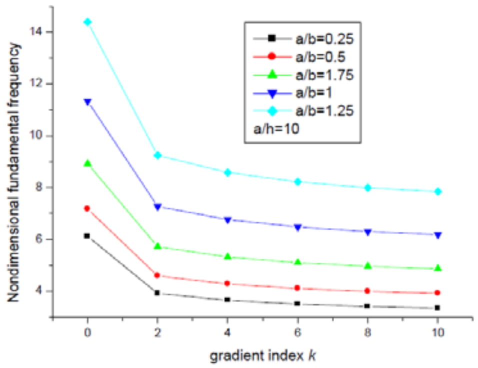

Fig. 3 Non-dimensional fundamental frequency $\bar{w} = \omega a^2 / h\sqrt{\rho_m / E_m}$ of Al/Al_2O_3 across power law index *k*

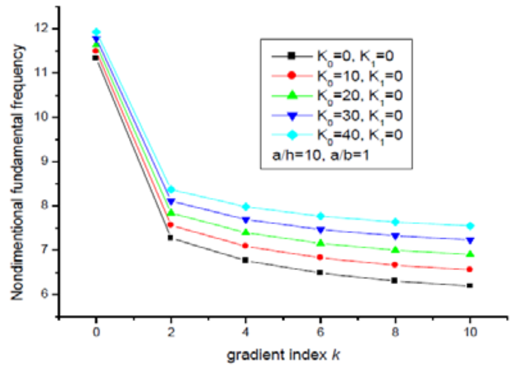

Fig. 4 Non-dimensional fundamental frequency $\bar{w} = \omega a^2 / h\sqrt{\rho_m / E_m}$ of Al/Al_2O_3 across power law index resting on the Winkler foundation

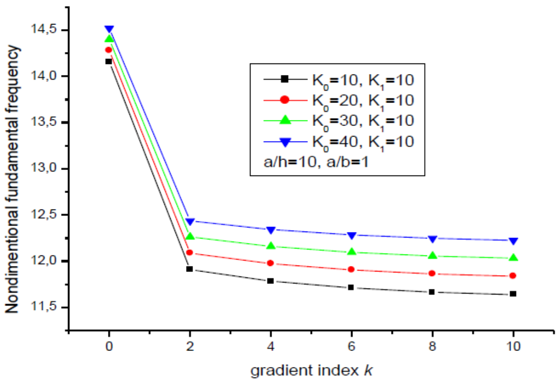

Fig. 5 Non-dimensional fundamental frequency $\bar{w} = \omega a^2 / h\sqrt{\rho_m / E_m}$ of Al/Al_2O_3 across power law index resting on the elastic foundation

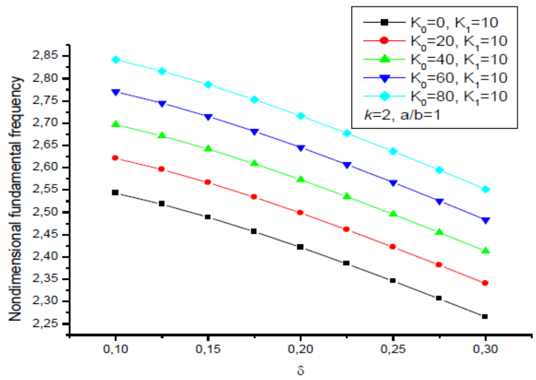

Fig. 6 Non-dimensional fundamental frequency $\bar{w} = \omega b^2 \sqrt{SH / A} / \pi^2$ of Al/Al_2O_3 across thickness to length ratio ($\delta = h / a$) resting on the elastic foundation

Figs. 4 and 5 displays the variation of the non-dimensional fundamental frequency $\bar{w} = \omega a^2 / h\sqrt{\rho_m / E_m}$ of Al/Al_2O_3 functionally graded square plates with respect to power law index *k* (*k*=0 to 10) resting on the Winkler and Winkler-Pasternak foundation respectively. It can be observed that the frequencies increase with the increase of the foundation parameters.

In Fig. 6, the variations of non-dimensional fundamental frequencies $\bar{w} = \omega b^2 \sqrt{SH / A} / \pi^2$ of simply supported Al/Al_2O_3 functionally graded square plates with respect to thickness to length ratio ($\delta = h / a$) are plotted. It is seen from the figure that, increasing value of Winkler coefficient of foundation causes to increase in the fundamental frequency.

5. Conclusions

In this work, an efficient new refined shear deformation theory based on the neutral surface concept was effectively used to study extensively the free vibration analysis of an FG simply-supported plate resting on the elastic foundations using analytical procedure. Equilibrium equations are obtained using Hamilton's principle. The Navier method is used for the analytical solutions of the functionally graded plate with simply supported boundary conditions. It was demonstrated that the present solution is highly efficient for exact analysis of the vibration of FG rectangular plates on the elastic foundations. Parametric studies for varying of the power low index, the foundation stiffness parameters, the aspect and side-to-thickness ratio are discussed and demonstrated through illustrative numerical examples. The present findings will be a useful benchmark for evaluating other analytical and numerical methods

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