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Approximate solution of fuzzy quadratic Riccati differential equations

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Abstract. This paper targets to investigate the solution of fuzzy quadratic Riccati differential equations with various types of fuzzy environment using Homotopy Perturbation Method (HPM). Fuzzy convex normalized sets are used for the fuzzy parameter and variables. Obtained results are depicted in term of plots to show the efficiency of the proposed method.

Keywords: fuzzy quadratic Riccati differential equation; fuzzy number; triangular fuzzy number; homotopy perturbation method

1. Introduction

Fuzzy differential equations have been expeditiously growing in recent years. Chang and Zadeh (1972) first introduced the concept of a fuzzy derivative, followed by Dubois and Prade (1982) who defined and used the extension principle in their approach. The fuzzy differential equation and fuzzy initial value problem are studied by Kaleva (1987, 1990) and Seikkala (1987). Numerical method for solving fuzzy differential equations is introduced by Ma *et al*. (1999) by the standard Euler method. Bede (2008) has also been described the numerical solution of fuzzy differential equations in his note in an excellent way. Chakraverty and Nayak (2012) implemented fuzzy finite element method for solving uncertain heat conduction problems. Recently Tapaswini and Chakraverty and Nayak (2012) have proposed a new method to solve fuzzy initial value problem Different authors developed various other methods to solve fuzzy differential equations. Here we have considered Homotopy Perturbation Method (HPM) to solve fuzzy quadratic Riccati differential equation with different cases as it has many important applications in the analysis and design of linear periodic control systems.

Crisp quadratic Riccati differential equation is solved by various authors using different approximation method as reported here. Biazar and Eslami (2010) applied differential transform method to solve the equation. Tan and Abbasbandy (2008) discussed the solution of the said differential equation by the homotopy analysis method. Batiha (2012) proposed a numeric analytic method for approximating the quadratic Riccati differential equation. HPM is also used by Abasbandy (2006a,b) to find the solution of quadratic differential equation. Aminikhah and Hemmatnezhad (2010) have also adopted new form of homotopy perturbation method for solving

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quadratic Riccati differential equation.

In the above mentioned papers the parameter and variable are considered as crisp. The uncertainty in model of quadratic Riccati differential equation makes the problem more difficult to handle.

But in actual practice the variables and coefficients in the differential equations are not crisp (fixed). Those variables and coefficients are obtained by either some experiment or by experience. As such the coefficients and the variables may be used in interval or in fuzzy sense. So, we need to solve ordinary and partial differential equations accordingly. Recently, Tapaswini and Chakraverty (2013) applied homotopy perturbation method for solving fuzzy $n-$th order fuzzy differential equations. As discussed above here we have applied the homotopy perturbation method for the solution of fuzzy quadratic Riccati differential equation so, related papers about the methods are cited as below.

The Homotopy Perturbation Method (HPM) was first developed by Ji-Huan He (1999, 2000) and many author's applied this method to solve linear and nonlinear functional equations in scientific and engineering problems with crisp parameters. The solution is considered as the sum of infinite series, which converges rapidly to accurate solutions. Using the Homotopy technique in topology, a homotopy is constructed with an embedding parameter which is considered as a "small parameter". Accordingly, this method is continuously deforming a difficult problem to a simple form which is easy to solve.

In this paper, new vista has been attempted when the uncertainty is considered in term of fuzzy or in interval for the solution of quadratic Riccati differential equation. One may use the variable and parameter in the differential equation itself as fuzzy/interval or only the initial condition as fuzzy or both. For the present investigation the fuzzy quadratic Riccati differential equation is explained with both coefficients and initial value as fuzzy. There exists various types of fuzzy numbers to handle the fuzzy initial value problem but here only the triangular fuzzy number is considered for the analysis.

In the following sections preliminaries is first discussed followed by homotopy perturbation method with crisp initial value. Then numerical implementation of HPM for fuzzy quadratic Riccati differential equation with various fuzzy conditions lastly the conclusions is drawn.

2. Preliminaries

We begin this section with defining some definitions which are used throughout this paper (Zimmermann 2001, Jaulin *et al*. 2001).

2.1 Definition

2.1.1 Fuzzy number

A fuzzy number $\tilde{U}$ is a convex normalised fuzzy set $\tilde{U}$ of the real line R such that

$$\{\mu_{\tilde{U}}(x) : R \to [0,1], \forall\, x \in R\}$$

where, $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set and it is piecewise continuous.

2.1.2 Triangular Fuzzy Number (TFN)

A triangular fuzzy number $\tilde{U}$ is a convex normalized fuzzy set $\tilde{U}$ of the real line R such t that

i. there exists exactly one $x_0 \in R$ with $\mu_{\tilde{U}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{U}$), where $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set,

ii. $\mu_{\tilde{U}}(x)$ is piecewise continuous.

We denote an arbitrary triangular fuzzy number as $\tilde{U} = (a_1, a_2, a_3)$. The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$ is then defined as follows

$$\mu_U(x) = \begin{cases} 0, & x \le a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \le x \le a_3 \\ 0, & x \ge a_3 \end{cases}$$

Triangular fuzzy number $\tilde{U}$ can be represented with an ordered pair of functions through $r-$cut approach *i.e* $[\underline{u}(r), \overline{u}(r)] = [(a_2 - a_1)r + a_1, -(a_3 - a_2)r + a_3]$ where $r \in [0,1]$.

This satisfies the following requirements

i. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0,1]$,

ii. $\overline{u}(r)$ is a bounded right continuous non-increasing function over $[0,1]$,

iii. $\underline{u}(r) \le \overline{u}(r), \quad 0 \le r \le 1$.

2.1.3 Fuzzy arithmetic

For any two arbitrary fuzzy number $\tilde{x} = [\underline{x}(r), \overline{x}(r)]$, $\tilde{y} = [\underline{y}(r), \overline{y}(r)]$ and scalar k, we have the fuzzy arithmetic as follows

i. $\tilde{x} = \tilde{y}$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$.

ii. $\tilde{x} + \tilde{y} = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$.

iii. $\tilde{x} \times \tilde{y} = \begin{bmatrix} \min\left(\underline{x}(r) \times \underline{y}(r), \underline{x}(r) \times \overline{y}(r), \overline{x}(r) \times \underline{y}(r), \overline{x}(r) \times \overline{y}(r)\right), \\ \max\left(\underline{x}(r) \times \underline{y}(r), \underline{x}(r) \times \overline{y}(r), \overline{x}(r) \times \underline{y}(r), \overline{x}(r) \times \overline{y}(r)\right) \end{bmatrix}$.

iv. $k\tilde{x} = \begin{cases} [k\overline{x}(r), k\underline{x}(r)], k < 0 \\ [k\underline{x}(r), k\overline{x}(r)], k \ge 0 \end{cases}$.

2.2 Lemma 1

If $u(t) = (x(t), y(t), z(t))$ is triangular number valued function and if u is Hukuhara differentiable, then $u' = (x', y', z')$ (Bede (2008)).

By using this property, if we intend to solve the FIVP

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

with $x_0 = (\underline{x}_0, x_0^1, \overline{x}_0) \in R, x(t) = (\underline{u}, u^1, \overline{u}) \in R$ and $f:[t_0, t_0 + a] \times R \to R,\quad f(t(\underline{u}, u', \overline{u})) = (\underline{f}(t, \underline{u}, u', \overline{u}),\ f^1(t, \underline{u}, u^1, \overline{u}), \overline{f}(t, \underline{u}, u^1, \overline{u})$ we can translate it into the system of ODEs

$$\begin{cases} \underline{u}' = \underline{f}(t, \underline{u}, u^1, \overline{u}), \\ u^{1'} = f^1(t, \underline{u}, u^1, \overline{u}) \\ \overline{u}' = \overline{f}(t, \underline{u}, u^1, \overline{u}) \\ \underline{u}(0) = \underline{x}_0, u^1(0) = x_0^1, \overline{u}(0) = \overline{x}_0. \end{cases} \tag{2}$$

2.3 Theorem 1 (Bede (2008))

Let us consider the FIVP (2) with $x_0 = (\underline{x}_0, x_0^1, \overline{x}_0)$ and $f:[t_0, t_0 + a] \times R \to R,\ f(t, (\underline{u}, u^1, \overline{u})) = (\underline{f}(t, \underline{u}, u^1, \overline{u})), f^1(t, \underline{u}, u^1, \overline{u}), \overline{f}(t, \underline{u}, u^1, \overline{u})$ such that $\underline{f}, f^1, \overline{f}$ are Lipschitz continuous (real-valued) functions.

Then the solution of (1) is triangular-valued function $x(t) = (\underline{u}(t), u^1(t), \overline{u}(t)) : [t_0, t_0 + a] \to R$ and the problem (1) is equivalent to problem (2).

3. Homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation of the form.

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{3}$$

With the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad t \in \Gamma \tag{4}$$

where, A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain $\boldsymbol{\Omega}$. A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Therefore Eq. (3) may be written as follows

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega \tag{5}$$

By the homotopy technique, we construct a homotopy $U(r,p) : \Omega \times [0,1] \to R$, which satisfies

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{6}$$

or

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{7}$$

where, $r \in \Omega$ and $p \in [0,1]$ is an imbedding parameter, v_0 is an initial approximation of Eq. (3).

Hence, it is obvious that

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{8}$$

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{9}$$

and the changing process of p from 0 to 1, is just that of $U(r, p)$ from $v_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(U) - L(v_0)$, $A(U) - f(r)$ are called homotopic. Applying the perturbation technique (He 1999, 2000), due to the fact that $0 \le p \le 1$ can be considered as a small parameter, we can assume that the solution of Eqs. (6) or (7) can as a power series in p as follows

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{10}$$

when $p \to 1$, Eqs. (6) or (7) corresponds to Eqs. (5) and (10) becomes the approximate solution of Eq. (5), i.e.

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{11}$$

The convergence of the series (11) has been proved in He (1999, 2000).

4. Numerical Implementation

4.1 Case 1

Let us consider the fuzzy quadratic Riccati differential equation

$$\frac{d}{dt}\tilde{y}(t) = 2\tilde{y}(t) - \tilde{y}^2(t) + 1 \tag{12}$$

with fuzzy initial condition in the term of triangular fuzzy number viz. $\tilde{y}(0) = (0.1, 0.2, 0.3)$. Using $r-$ cut triangular fuzzy initial condition becomes

$$\tilde{y}(0;r) = [0.1r + 0.1, 0.3 - 0.1r], \quad 0 \le r \le 1$$

Here $\tilde{y}(t)$ is a fuzzy function of t.

The exact solution of Eq. (12) with crisp initial condition that is $y(0) = 0.2$, is found to be

$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{5\sqrt{2}-4}{5\sqrt{2}+4}\right)\right)$. Corresponding plot is given in Fig 1.

Expanding $y(t)$ using Taylor expansion about $t = 0$ gives

$$y(t)=\frac{1}{5}+\frac{34}{25}t+\frac{136}{125}t^2-\frac{68}{1875}t^3-\frac{7072}{9375}t^4-\frac{21488}{46875}t^5+\frac{163744}{703125}t^6+\frac{11459632}{24609375}t^7+\frac{16217728}{123046875}t^8-\frac{1331830592}{5537109375}t^9-\cdots.$$

Using Lemma 1, the fuzzy Riccati differential equation may be reduced to set of ordinary differential equations

$$\frac{d\underline{y}(t;r)}{dt}=2\underline{y}(t;r)-\bar{y}^2(t;r)+1 \tag{13}$$

$$\frac{d\bar{y}(t;r)}{dt}=2\bar{y}(t;r)-\underline{y}^2(t;r)+1 \tag{14}$$

We can readily construct a simple homotopy for Eqs. (13) and (14) respectively which satisfies

$$(1-p)\frac{d\underline{y}(t;r)}{dt}+p\left[\frac{d\underline{y}(t;r)}{dt}-2\underline{y}(t;r)+\bar{y}^2(t;r)-1\right]=0 \tag{15}$$

$$(1-p)\frac{d\bar{y}(t;r)}{dt}+p\left[\frac{d\bar{y}(t;r)}{dt}-2\bar{y}(t;r)+\underline{y}^2(t;r)-1\right]=0 \tag{16}$$

Next, we can assume the solution of Eqs. (15) or (16) as a power series expansion in p as

$$\underline{y}(t;r)=\underline{y}_0(t;r)+p\underline{y}_1(t;r)+p^2\underline{y}_2(t;r)+\cdots \tag{17}$$

$$\bar{y}(t;r)=\bar{y}_0(t;r)+p\bar{y}_1(t;r)+p^2\bar{y}_2(t;r)+\cdots \tag{18}$$

where, $\underline{y}_i(t;r)$ and $\bar{y}_i(t;r)$ for $i=0,1,2,3,\cdots$ are functions yet to be determined. Substituting Eqs. (17) and (18) into Eqs. (15) and (16) respectively, and equating the terms with the identical powers of p, we have

$$\begin{aligned}
&p^0:\begin{cases}\dfrac{d\underline{y}_0(t;r)}{dt}=0\\[2mm] \dfrac{d\bar{y}_0(t;r)}{dt}=0\end{cases}\\
&p^1:\begin{cases}\dfrac{d\underline{y}_1(t;r)}{dt}-2\underline{y}_0(t;r)+\bar{y}_0^2(t;r)-1=0\\[2mm] \dfrac{d\bar{y}_1(t;r)}{dt}-2\bar{y}_0(t;r)+\underline{y}_0^2(t;r)-1=0\end{cases}\\
&p^2:\begin{cases}\dfrac{d\underline{y}_2(t;r)}{dt}-2\underline{y}_1(t;r)+2\bar{y}_0(t;r)\bar{y}_1(t;r)=0\\[2mm] \dfrac{d\bar{y}_2(t;r)}{dt}-2\bar{y}_1(t;r)+2\underline{y}_0(t;r)\underline{y}_1(t;r)=0\end{cases}
\end{aligned} \tag{19}$$

and so on.

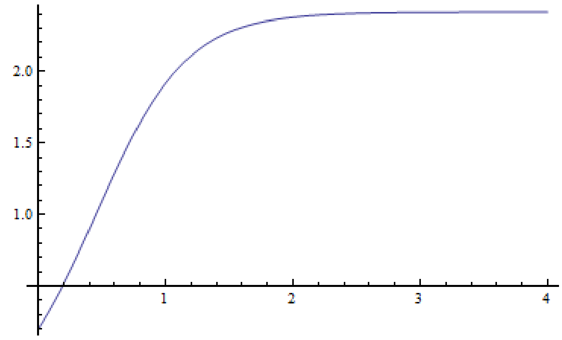

Fig. 1 The exact solution

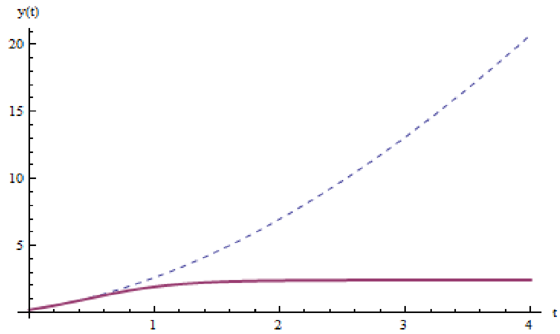

Fig. 2 The exact solution (-), versus the HPM solution (- -)

Choosing initial approximation $\underline{y}_0(t;r)$ and $\overline{y}_0(t;r)$ and applying the inverse operator L_t^{-1} (which is the inverse of the operator $L_t = \frac{d}{dt}$) on both sides of every Eq. (19) one may obtain the following equations

$$p^0 : \begin{cases} \underline{y}_0(t;r) = \underline{y}(0;r), \\ \overline{y}_0(t;r) = \overline{y}(0;r), \end{cases}$$

$$p^1 : \begin{cases} \underline{y}_1(t;r) = L_t^{-1}\left(2\underline{y}_0(t;r) - \overline{y}_0^2(t;r) + 1\right), \\ \overline{y}_1(t;r) = L_t^{-1}\left(2\overline{y}_0(t;r) - \underline{y}_0^2(t;r) + 1\right), \end{cases}$$

$$p^2 : \begin{cases} \underline{y}_2(t;r) = L_t^{-1}\left(2\underline{y}_1(t;r) - 2\overline{y}_0(t;r)\overline{y}_1(t;r)\right), \\ \overline{y}_2(t;r) = L_t^{-1}\left(2\overline{y}_1(t;r) - 2\underline{y}_0(t;r)\underline{y}_1(t;r)\right), \end{cases}$$

then we obtain

$$\underline{y}_0(t;r) = 0.1r + 0.1 \ ,$$

$$\overline{y}_0(t;r) = 0.3 - 0.1r \ ,$$

$$\underline{y}_1(t;r) = (1.11 + 0.26r - 0.01r^2)t \ ,$$

$$\overline{y}_1(t;r) = (1.59 - 0.22r - 0.01r^2)t \ ,$$

$$\underline{y}_2(t;r) = (0.633 + 0.485r - 0.029r^2 - 0.001r^3)t^2 \ ,$$

$$\overline{y}_2(t;r) = (1.479 - 0.357r - 0.0359r^2 + 0.001r^3)t^2$$

Proceeding in this manner, the rest of the components $\tilde{y}_i(t;r) = [\underline{y}_i(t;r), \overline{y}_i(t;r)]$ can be obtained and the third term approximate solution of Eqs. (13) and (14) in finite series form is given by

$$\begin{aligned} \underline{y}(t;r) &\cong \underline{y}_0(t;r) + \underline{y}_1(t;r) + \underline{y}_2(t;r) \\ &= (0.1r + 0.1) + (1.11 + 0.26r - 0.01r^2)t + (0.633 + 0.485r - 0.029r^2 - 0.001r^3)t^2 \end{aligned}$$

$$\bar{y}(t;r) \cong \bar{y}_0(t;r) + \bar{y}_1(t;r) + \bar{y}_2(t;r)$$
$$= (0.3 - 0.1r) + (1.59 - 0.22r - 0.01r^2)t + (1.479 - 0.357r - 0.0359r^2 + 0.001r^3)t^2$$

Figs. 3 to 7 depict triangular fuzzy solution for $t = 0, 1, 2, 3$ and 4 respectively. Where, dashed and solid line corresponds to lower and upper bounds of the fuzzy solutions. The thick line represents the centre (crisp) solution for $r = 1$, with the fuzzy initial condition. This is found to be exactly equal with the crisp Riccati differential equation for crisp initial condition.

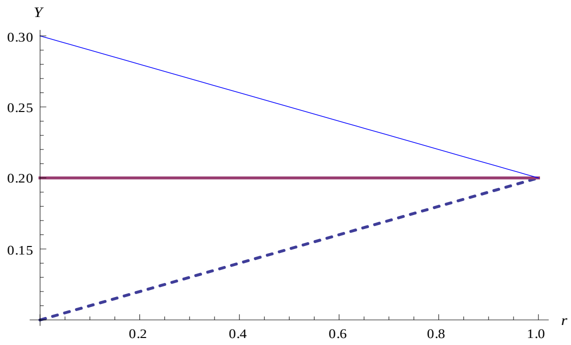

Fig. 3 Solution plots for $t = 0$

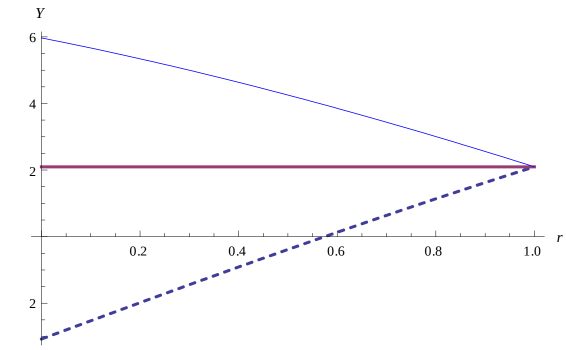

Fig. 4 Solution plots for $t = 1$

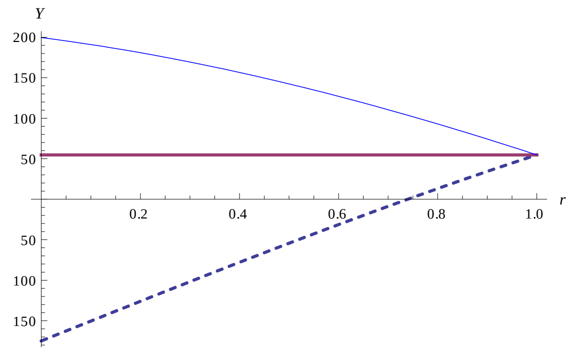

Fig. 5 Solution plots for $t = 2$

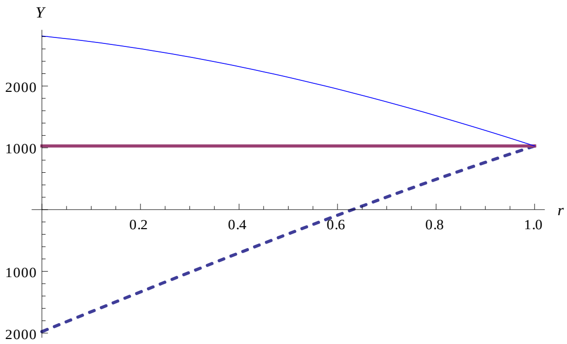

Fig. 6 Solution plots for $t = 3$

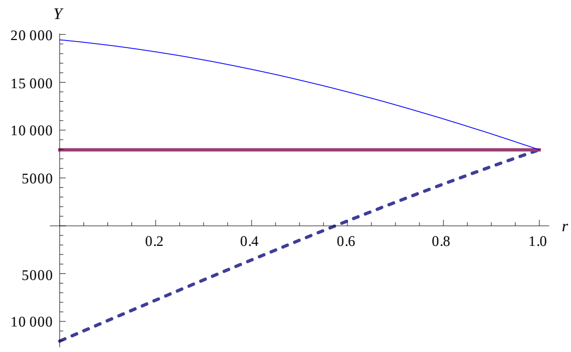

Fig. 7 Solution plots for $t = 4$

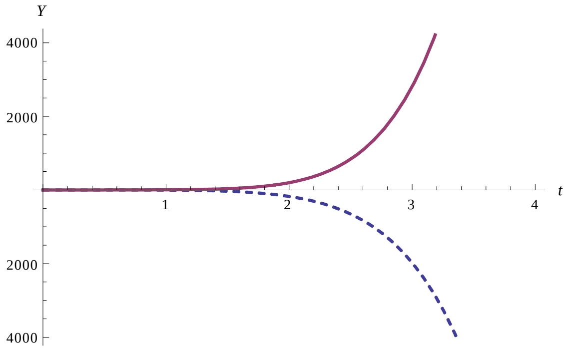

Fig. 8 Solution plots for $r = 0$

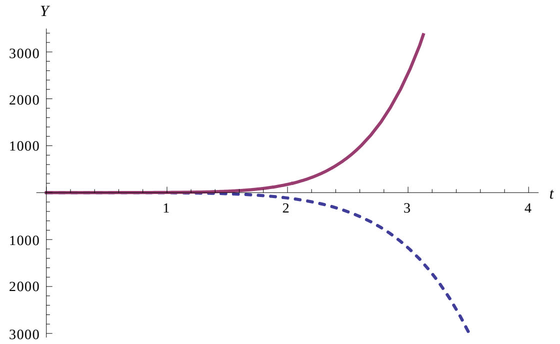

Fig. 9 Solution plots for $r = 0.25$

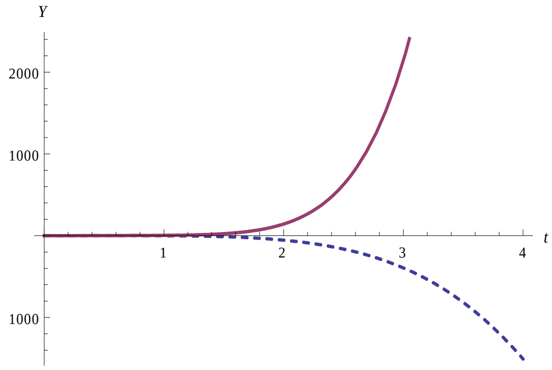

Fig. 10 Solution plots for $r = 0.5$

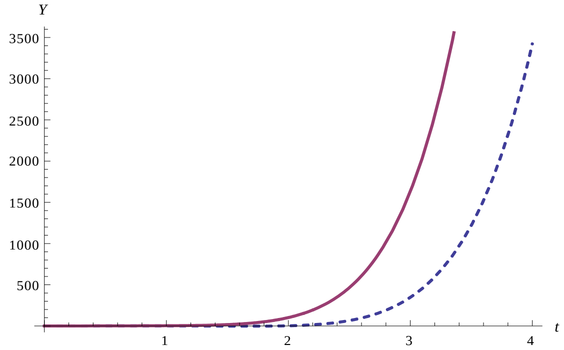

Fig. 11 Solution plots for $r = 0.75$

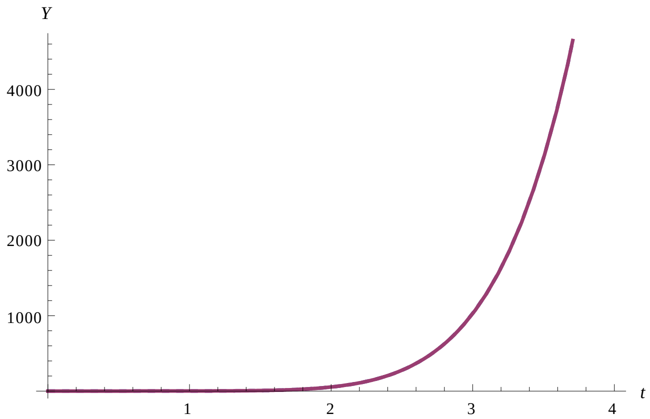

Fig. 12 Solution plots for $r = 1$

One may see that the solution plot for $r = 0$, 0.25, 0.5 and 0.75 as shown in Figs. 8 to 11 respectively give the interval solution. On the other hand for $r = 1$ Fig. 10 gives the crisp solution. Here a dashed and solid line represents the lower and upper bound of the interval solution respectively. It is interesting to note that the lower and upper bounds are same for $r = 1$, which is exactly agrees with crisp solution.

4.2 Case 2

Now consider the fuzzy quadratic Riccati differential equation

$$\begin{aligned}&\frac{d}{dt}(\underline{y}(t;r), y^C(t;r), \bar{y}(t;r)) = (1.9,\,2,\,2.1)(\underline{y},(t;r), y^C(t;r), \bar{y}(t;r))\\ &-(\underline{y}(t;r), y^C(t;r), \bar{y}(t;r))^2 + (0.9, 1, 1.1)\end{aligned} \tag{20}$$

where, coefficient is in term of triangular fuzzy number with crisp initial condition $y(0) = (0.2)$.

Using HPM method, we obtain the seven-term approximate solution of Eq. (20) in finite series form is given by

$$\begin{aligned}\underline{y}(t;r) \cong\ & 0.2 + (1.24 + 0.12r)t + 0.006(5.23086 + r)(28.1025 + r)t^2\\ &+ 0.0002(-1.11502 + r)(1550.04 + 25.4484r + r^2)t^3\\ &+ 5\times10^{-6}(-2.95147 + r)(136.419 + r)(575.725 - 14.134r + r^2)t^4\\ &+ 10^{-7}(-2.11254 + r)(441.982 - 24.8716r + r^2)(9860.5 - 6.68229r + r^2)t^5\\ &+ 1.66667\times10^{-9}(-36.0621 + r)(-19.8172 + r)(-0.480014 + r)(410.276 + r)\\ &(993.881 - 4.58338r + r^2)t^6 + 2.38096\times10^{-11}(-143.607 + r)(-10.584 + r)\\ &(0.0924811 + r)(22.782 - 235.487r + r^2)(570.002 + 9.9207r + r^2)t^7\end{aligned}$$

$$\begin{aligned}\bar{y}(t;r) \cong\ & 0.2 + (1.48 - 0.12r)t + 0.006(-30.1025 + r)(-7.23086 + r)t^2\\ &- 0.0002(-0.884979 + r)\,(1604.94 - 29.4484r + r^2)t^3\\ &+ 5\times10^{-6}(-138.419 + r)(0.951471 + r)(551.458 + 10.134r + r^2)t^4\\ &- 10^{-7}(0.112545 + r)(9851.13 + 2.68234r + r^2)(396.239 + 20.8716r + r^2)t^5\\ &+ 1.66667\times10^{-9}(-412.276 + r)(-1.51998 + r)(17.8173 + r)(34.0621 + r)988.712\\ &+ 0.583368r + r^2)t^6 - 2.38096\times10^{-11}(-2.09248 + r)(8.58404 + r)(141.608 + r)\\ &(593.843 - 13.9208r + r^2)(22314.9231.487r + r^2)t^7\end{aligned}$$

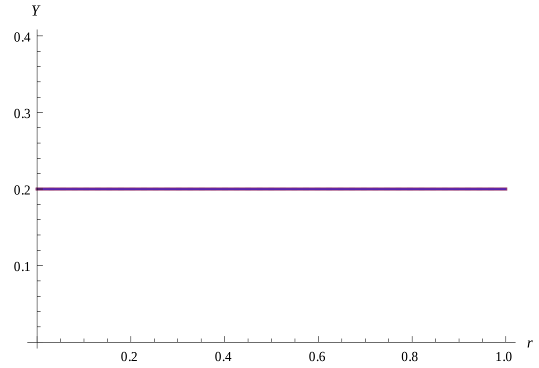

Fig. 13 Solution plots for $t = 0$

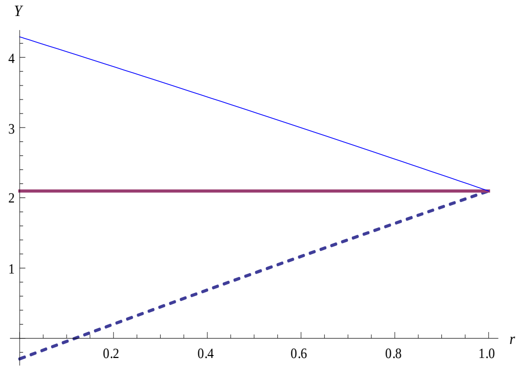

Fig. 14 Solution plots for $t = 1$

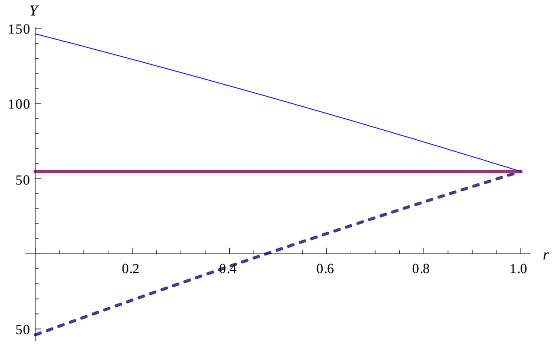

Fig. 15 Solution plots for $t = 2$

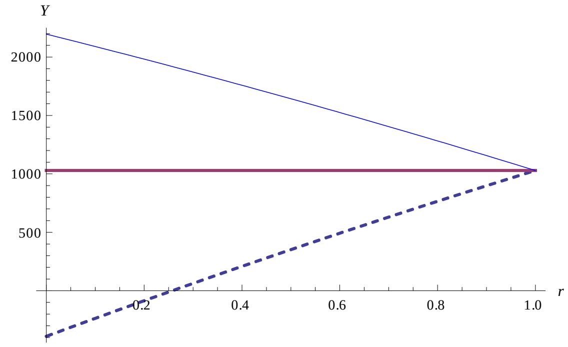

Fig. 16 Solution plots for $t = 3$

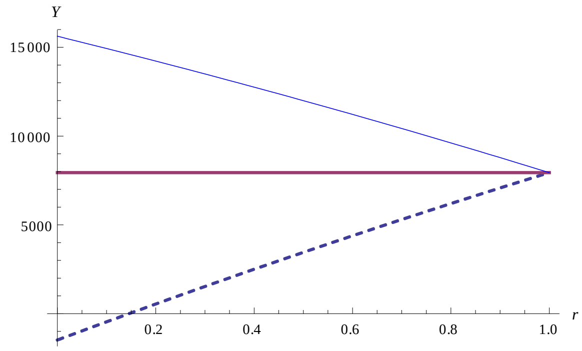

Fig. 17 Solution plots for $t = 4$

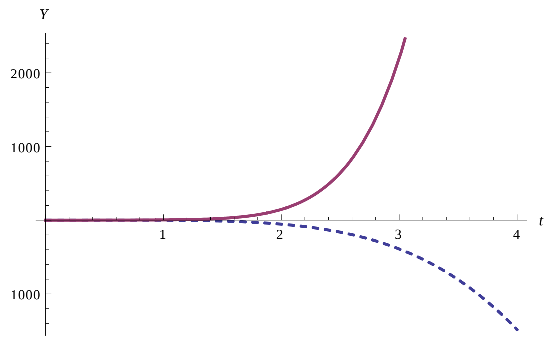

Fig. 18 Solution plots for $r = 0$

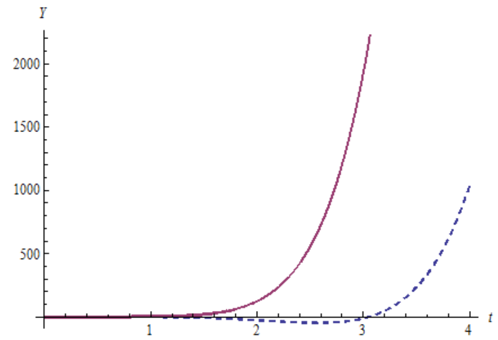

Fig. 19 Solution plots for $r = 0.25$

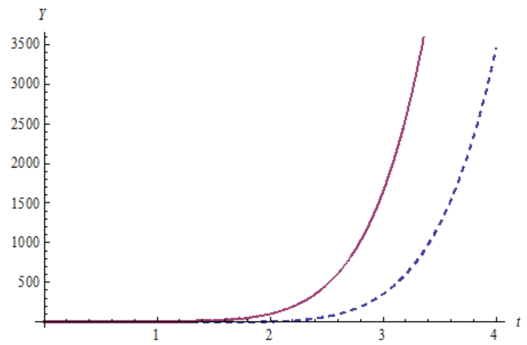

Fig. 20 Solution plots for $r = 0.5$

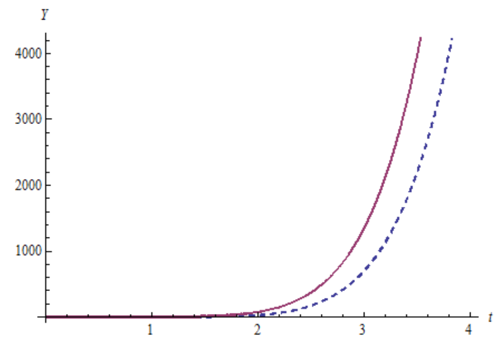

Fig. 21 Solution plots for $r = 0.75$

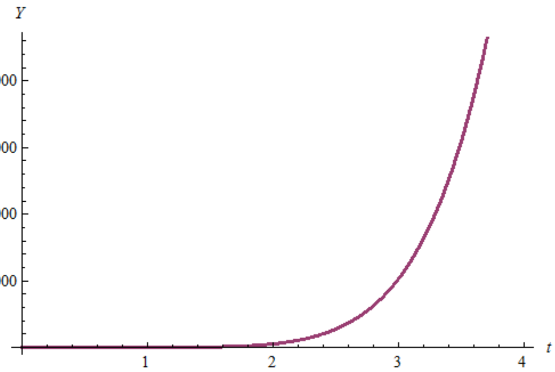

Fig. 22 Solution plots for $r=1$

For this case Figs. 13 to 17 depict fuzzy solution for t = 0, 1, 2, 3 and 4 respectively.

The solution plot for r = 0, 0.25, 0.5 and 0.75 as shown in Figs. 18 to 21 which give the interval solution. On the other hand for $r=1$ Fig. 22 gives the crisp solution.

4.3 Case 3

Let us now consider the fuzzy quadratic Riccati differential equation as

$$\frac{d}{dt}(\underline{y}(t;r), y^C(t;r), \bar{y}(t;r)) = (1.9,\, 2,\, 2.1)(\underline{y}(t;r), y^C(t;r), \bar{y}(t;r)) \\ -(\underline{y}(t;r), y^C(t), \bar{y}(t;r))^2 + (0.9\,,1,1.1) \tag{21}$$

where, both coefficients and initial condition are triangular fuzzy number. The initial condition taken as $\tilde{y}(0) = (0.1,0.2,0.3)$.

Using HPM method, one may get approximate solution of the discussed fuzzy differential equation as

$$\begin{aligned}
\underline{y}(t;r) &= (0.1r+0.1) + (1+0.36r)t - 0.018(-37.9684+r)(0.635032+r)t^2 \\
&\quad - 0.0018(-11.1853+r)(-1.03853+r)(50.3349+r)t^3 \\
&\quad + 0.000045(-234.634+r)(-6.10268+r)(-1.69946+r)(19.1029+r)t^4 \\
&\quad + 2.5\times10^{-6}(-60.8253+r)(-4.44944+r)(-1.37686+r)(9.73525+r)(202.361+r)t^5 \\
&\quad - 4.5\times10^{-8}(-877.774+r)(-29.7664+r)(-3.7069+r)(-0.82647+r)(5.60491+r) \\
&\quad (65.1352+r)t^6 - 1.92857\times10^{-9}(-179.548+r)(-18.6442+r)(-3.01451+r) \\
&\quad (-0.635696+r)(3.4523+r)(32.1284+r)(707.039+r)t^7 \\
\bar{y}(t;r) &= 0.3 - 0.1r + (1.72-0.36r)t - 0.018(-2.63503+r)(35.9684+r)t^2 \\
&\quad + 0.0018(-52.3349+r)(-0.961465+r)(9.1852+r)t^3
\end{aligned}$$

$$+\,0.000045(-21.1029+r)(-0.300534+r)(4.10266+r)(232.634+r)t^4$$
$$-\,2.7\times10^{-6}(-204.361+r)(-11.7353+r)(-0.623139+r)(2.44943+r)(58.8253+r)t^5$$
$$-\,4.5\times10^{-8}(-67.1352+r)(-7.60494+r)(-1.17353+r)\,(1.7069+r)(27.7664+r)$$
$$(875.774+r)t^6+1.92857\times10^{-9}(-709.039+r)(-34.1284+r)(-5.45236+r)(-1.3643+r)$$
$$(1.01451+r)(16.6442+r)(177.548+r)t^7$$

As in previous cases again, Figs. 23 to 27 depict fuzzy solution for t = 0, 1, 2, 3 and 4 respectively.

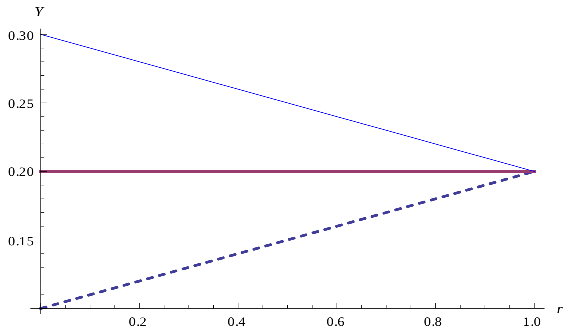

Fig. 23 Solution plots for $t=0$

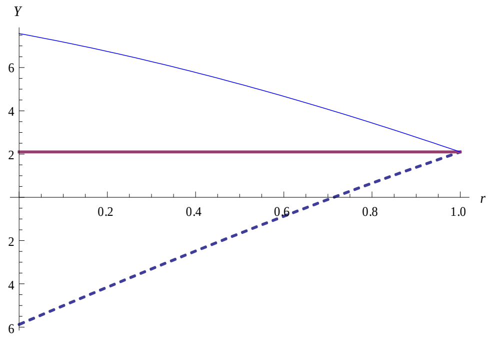

Fig. 24 Solution plots for $t=1$

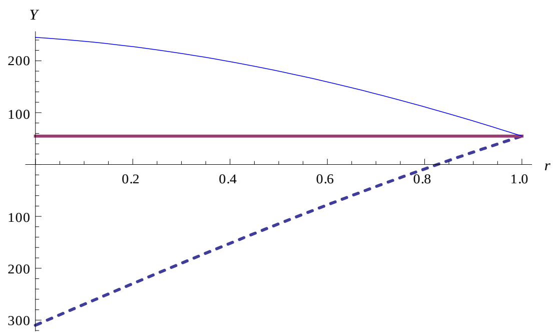

Fig. 25 Solution plots for $t=2$

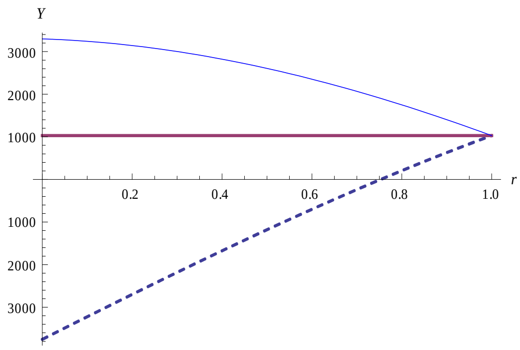

Fig. 26 Solution plots for $t=3$

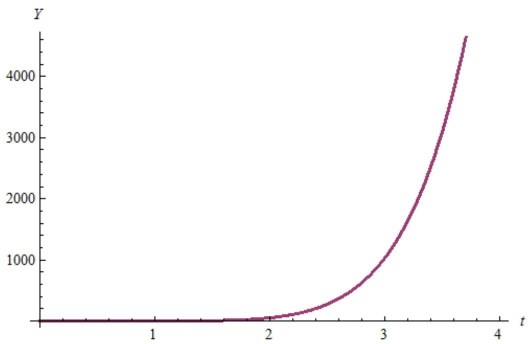

Fig. 27 Solution plots for $t=4$

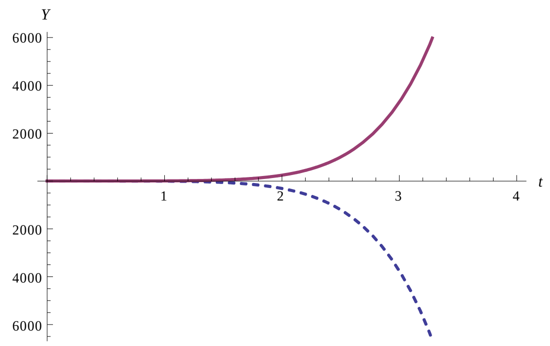

Fig. 28 Solution plots for $r = 0$

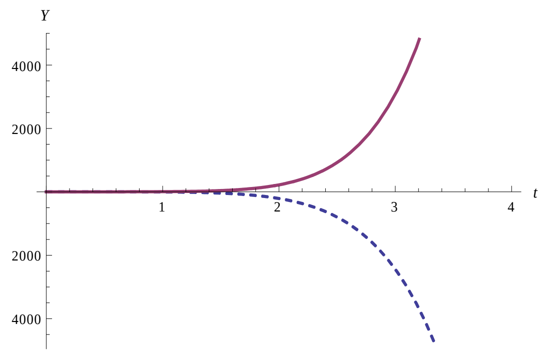

Fig. 29 Solution plots for $r = 0.25$

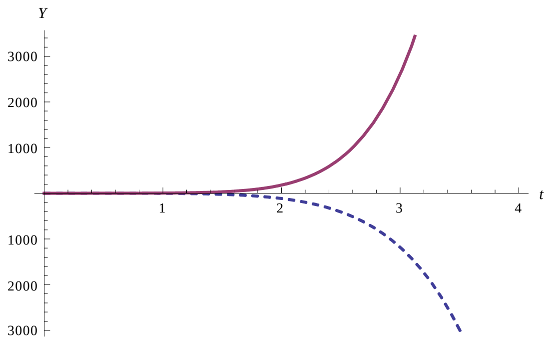

Fig. 30 Solution plots for $r = 0.5$

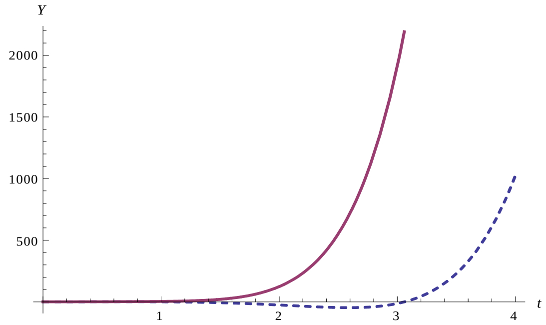

Fig. 31 Solution plots for $r = 0.75$

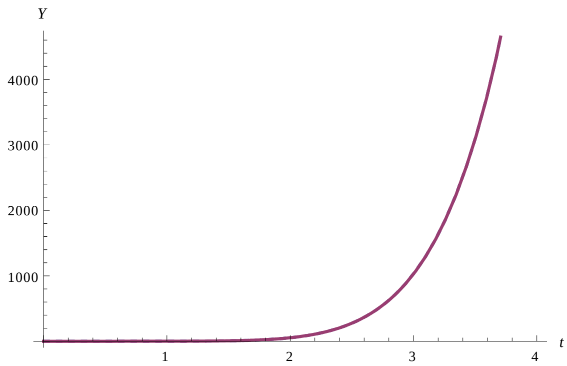

Fig. 32 Solution plots for $r = 1$

Figs. 28 to 31 depict the interval solution for this case. Fig. 32 shows the corresponding crisp solution ($r = 1$).

5. Conclusions

In this paper, HPM has been successful applied to find the solution of fuzzy quadratic Riccati differential equations. The solution obtained by HPM is an infinite series with appropriate initial condition, which in turn be expressed in a closed form i.e., the exact solution. The result shows that the HPM is a powerful mathematical tool to solve fuzzy quadratic Riccati differential equation. It is also a promising method to solve other nonlinear equation. The solutions obtained are shown graphically. In our work, we use Mathematica package to calculate the series obtained from HPM.

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References

Abbasbandy, S. (2006a), "Iterated He's homotopy perturbation method for quadratic Riccati differential equation", *Appl. Math. Comput.*, **175**(1), 581-589.

Abbasbandy, S. (2006b), "Homotopy perturbation method for quadratic Riccati differential equation and comparision with Adomian's decomposition method", *Appl. Math. Comput.*, **172**(1), 485-490.

Aminikhah, H. and Hemmatnezhad, M. (2010), "An efficient method for quadratic Riccati differential equation", *Commun. Nonlinear Sci. Numer. Simul.*, **15**(4), 835-839.

Batiha, B. (2012), "A numeric-analytic method for approximating quadratic Riccati differential equation", *Int. J. Appl. Math. Res.*, **1**(1), 8-16.

Bede, B. (2008), "Note on numerical solutions of fuzzy differential equations by predictor-corrector method", *Inform. Sciences*, **178**(7), 1917-1922.

Biazar, J. and Eslami, M. (2010), "Differential transform method for quadratic Riccati differential equation", *Int. J. Nonlinear Sci.*, **9**(4), 444-447.

Chang, S.L. and Zadeh, L.A. (1972), "On fuzzy mapping and control", *IEEE T. Syst. Man Cy.*, **2**(1), 30-34.

Chakraverty, S. and Nayak, S. (2012), "Fuzzy finite element method for solving uncertain heat conduction problems", *Coupled Syst. Mech.*, **1**(4), 345-360.

Dubois, D. and Prade, H. (1982), "Towards fuzzy differential calculus: Part 3 differentiation", *Fuzzy Set. Syst.*, **8**(3), 225-233.

He, J. H. (1999), "Homotopy perturbation technique", *Comput. Method. Appl. M.*, **178**(3-4), 257-262.

He, J. H. (2000), "A coupling method of homotopy technique and a perturbation technique for nonlinear problems", *Int. J. Nonlinear Mech.*, **35**(1), 37-43.

Jaulin, L., Kieffer, M., Didrit, O. and Walter, E. (2001), *Applied interval analysis*, Springer.

Kaleva, O. (1987), "Fuzzy differential equations", *Fuzzy Set. Syst.*, **24**(3), 301-317.

Kaleva, O. (1990), "The Cauchy problem for fuzzy differential equations", *Fuzzy Set. Syst.*, **35**(3), 389-396.

Ma, M., Friedman, M. and Kandel, A. (1999), "Numerical solutions of fuzzy differential equations", *Fuzzy Set. Syst.*, **105**(1), 133-138.

Seikkala, S. (1987), "On the fuzzy initial value problem", *Fuzzy Set. Syst.*, **24**(3), 319-330.

Tan, Y. and Abbasbandy, S. (2008), "Homotopy analysis method for quadratic Riccati differential equation", *Commun. Nonlinear Sci. Numer. Simul.*, **13**(3), 539-546.

Tapaswini, S. and Chakraverty, S. (2013), "Numerical solution of $n-$th order fuzzy linear differential equations by homotopy perturbation method", *Int. J. Comput. Appl.*, **64**(6), 5-10.

Tapaswini S. and Chakraverty S. (2012), "A new approach to fuzzy initial value problem by improved euler method", *Int. J. Fuzzy Inform. Eng.*, **4**(3), 293-312.

Zimmermann, H.J. (2001), *Fuzzy set theory and its application*, Kluwer Academic Publishers, Boston/ Dordrecht/London.