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# Anisotropic damage modelling of biaxial behaviour and rupture of concrete structures

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**Abstract.** This paper deals with damage induced anisotropy modeling for concrete-like materials. A thermodynamics based constitutive relationship is presented coupling anisotropic damage and elasticity, the main idea of the model being that damage anisotropy is responsible for the dissymmetry tension/compression. A strain written damage criterion is considered (Mazars criterion extended to anisotropy in the initial model). The biaxial behavior of a family of anisotropic damage model is analyzed through the effects of yield surface modifications by the introduction of new equivalent strains.

**Keywords:** constitutive relations; anisotropic damage; yield function; multiaxial behaviour.

## 1. Introduction

Regarding the ultimate behavior of reinforced and prestressed concrete structures, the prevision of oriented micro-cracks openings is of main importance for structural integrity assessment analyses (Bazant & Prat 1988, Carol, *et al.* 1992, Fichant, *et al.* 1999). It is even a key issue for multi-physics analyses such as in diffusion problems. In that sense, a mesoscopic approach using Continuum Damage Mechanics at the Representative Element Volume scale is a relevant tool to deal with large scale structures if loading induced damage anisotropy is handled (Lemaitre & Desmorat 2005). It allows representing the local loss of stiffness of the material and the strain localization zone representative of macroscopic cracks. Within the thermodynamics framework, the state damage variable representing the microcracks pattern may be a scalar or a tensorial quantity. It has been shown (Desmorat 2004) that anisotropy is responsible for the dissymmetry tension/compression. Then, a single damage variable is introduced for initially isotropic quasi-brittle materials. Dealing with induced anisotropy in cementious materials and with non-symmetric tension/compression behavior, the choice for a second order damage tensor (Cordebois & Sidoroff 1982, Ladevèze 1983, Murakami 1988) is a pragmatic choice made in this work with regard to robustness and numerical implementations. It is nevertheless important to point out that the 3D effects remain difficult to take into account within damage models. A coupling with plasticity is often required to gain some important features as of course permanent strains but also as confined features and responses.

The states of stresses being naturally multiaxial at the Gauss points level, a good approximation of the material response sustaining multiaxial states of stresses is required in Finite Element analyses.

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As already mentioned, works have then combined damage behavior for tensile loadings and plasticity-like behavior – eventually coupled with damage – for compressive or confined loadings (Yazdani & Schreyer 1990, Meschke & Lackner 1998, Gatuingt & Pijaudier-Cabot 2002, Jason, *et al.* 2006). The drawback of such approaches is the complexity of the models and their numerical implementation by dealing with multi-surfaces plasticity-damage modeling (the advantage is a large validity domain). Robustness is then difficult to ensure and making the modeling mesh independent (for instance by making the models nonlocal) also becomes a difficult task. On the other hand, damage models (with no plasticity) usually do not represent permanent strains, but they represent properly the monotonic softening response of materials, at least under low confinement conditions, often encountered in structural design when tension and shear are the main causes of structural failure. Not modeling plasticity reduces the model complexity and the numbers of materials parameters introduced. It allows also for considering more naturally criterion surfaces in the strains space, choices computationally efficient.

The choice is made in the present work to address such monotonic failure cases and to see wether damage models, anisotropic but with a limited numbers of material parameters, may be sufficient in this task. Concerning modeling, the tension/compression coupled to shear response usually needs an adequate expression for the equivalent stresses or strains used in the threshold or criterion function. Based on the works of Mazars (1984), Drucker-Prager (1952) and de Vree (1995), different strain based anisotropic damage criteria are proposed next and extended to nonlocal framework. Their influence at both material and structural levels is studied when ultimate behavior and rupture occur. In a first part, the initial thermodynamics damage model is recalled. The main numerical features of the new modelling are mentioned in a second step. A third part is dedicated to the bi-axial response at the material level, pointing out the numerical robustness through a classical benchmark and emphasizing the effects of the different expressions for the equivalent strains (for instance elasticity domains regarding the biaxial behavior of concrete at the material scale). At last, a structural example is presented.

## 2. Anisotropic modelling of concrete

The necessity to account for micro-cracks orientation in the description of the mechanical behavior of concrete naturally leads to the use of continuum anisotropic damage framework (Chaboche, 1979, Cordebois & Sidoroff 1982, Ladevèze 1983, Chow & Wang 1987, Murakami 1988, Halm & Dragon 1998, Lemaitre & Desmorat 2005, Badel, *et al.* 2007). To be computationally efficient, the expression of constitutive equations has to be coupled with a proper numerical algorithm for finite element analyses at the structural scale. From the numerical point of view, considering both a hydrostatic/deviatoric splitting (Papa & Talercio 1996) and a damage threshold based on an equivalent strain allows to avoid, at the Gauss point level, an iterative resolution of the constitutive equations and related evolution laws, even if implicitly discretized (Desmorat, *et al.* 2007). The efficiency of such an approach for the use of anisotropic damage in reinforced and pre-stressed concrete has been pointed out treating large scale structures.

The mesh dependency induced by strain softening at the local level is avoided by adopting integral nonlocal type regularization (Pijaudier-Cabot & Bazant 1987, Peerlings 1997). The transition between an homogeneous state of cracking and a macro-crack propagation is solved by adopting a scalar critical damage value $Dc$ close to unity beyond which the principal damages directions are kept fixed and allowing to pass from a rotating crack model towards a fixed crack one.

### 2.1. Elasticity coupled with anisotropic damage

Modelling micro-cracks initiation and growth at the Representative Volume Element scale within a macroscopic phenomenological framework needs the introduction of a thermodynamics variable. A choice has to be made concerning the damage kinematics, between scalar or tensorial representations. The easiest choice consists in using a scalar damage variable, representing an isotropic state of concrete degradation (Mazars 1984). This approach allows for the expression of efficient material models dealing with robust stress integration algorithms. In isotropic damage model, the notions of tensile damage and compressive damage are sometimes introduced. These are inconsistent with the idea of a state of variable representing a micro-cracking pattern. Large scale computation can be handled nevertheless with no information concerning the micro-cracks orientation. If the description of the micro-crack orientation is a major point, an anisotropic kinematics for the damage variable has to be introduced. Damage affects directly the elasticity law: the most natural approach could use a fourth order damage tensor (Chaboche 1979, Leckie & Onat 1981). It is possible to express a new thermodynamic variable or to directly use the elasticity operator as a variable which can be degradated (Krajcinovic 1985, Simo & Ju 1987, Ju 1989, Govindjee, *et al.* 1995, Meschke, *et al.* 1998). In such a case, the framework encounters strong limitations due to the difficulty to proceed robust numerical integration within a finite element code and to deal with adequate identification due to the large number of material parameters involved.

An alternative consists in using a symmetric orthotropic second order damage tensor as thermodynamic variable (Murakami & Ohno 1978, Cordebois & Sidoroff 1982, Chow & Wang 1987, Murakami 1988), valid for tensile, compressive or 3D any loading cases. The resulting elasticity operator has to be symmetric, depending classically on the principle of strain or energy equivalence used to define an effective stress (the stress 'effectively' acting on the resisting part of the material). Difficulty remains in the general case when dealing with effective stresses. For instance, they must be symmetric and preferably independent from the elasticity parameters (Lemaitre 2002, Lemaitre & Desmorat 2005). Difficulty also arises when accounting for complete stiffness recovery when passing from tension to compression (Ladevèze 1983).

In the present study, only partial stress recovery is introduced (recovery of the bulk modulus, not of the shear modulus). A single but tensorial damage variable is used describing the non-symmetric behaviour of concrete in tension and in compression. According to the general expression (Ladevèze 1983, Lemaitre & Desmorat 2005) insuring continuity of the stress-strain path whatever the loadings, the thermodynamics potential (Gibbs free energy) takes the following form:

$$\rho\psi^* = \frac{1+\nu}{2E} Tr[(\mathbf{1}-\boldsymbol{D})^{-1/2}\cdot\sigma^D\cdot(\mathbf{1}-\boldsymbol{D})^{-1/2}\cdot\sigma^D] + \frac{1-2\nu}{E}\left[\frac{\langle Tr\boldsymbol{\sigma}\rangle_+^2}{1-Tr\boldsymbol{D}} + \langle Tr\boldsymbol{\sigma}\rangle_-^2\right] \tag{1}$$

where $E$ and $\nu$ are the elasticity parameters, $\rho$ the density, $\sigma$ the Cauchy stress and $D$ the damage tensor. The notation $(\cdot)^D$ indicates the deviatoric part of a tensor and $(\mathbf{1}-\boldsymbol{D})^{-1/2}$ is gained from the diagonal form of $(\mathbf{1}-\boldsymbol{D})$ as $(\mathbf{1}-\boldsymbol{D})_{diag} = \boldsymbol{P}^{-1}\cdot(\mathbf{1}-\boldsymbol{D})\cdot\boldsymbol{P}$ and $(\mathbf{1}-\boldsymbol{D})^{-1/2} = \boldsymbol{P}\cdot(\mathbf{1}-\boldsymbol{D})_{diag}^{-1/2}\cdot\boldsymbol{P}^{-1}\cdot\mathbf{1}$ is the identity tensor and $\boldsymbol{P}$ the change-of-coordinate matrix. The notation $\langle x\rangle_+ = \max(0,x)$ stands for the positive part of the scalar $x$ and $\langle x\rangle_- = \min(0,x)$ stands for its negative part. The state variables are the Cauchy stress tensor and the second order anisotropic damage tensor.

The state laws are obtained by derivation of the potential with respect to the thermodynamics variables. The elasticity law coupled with anisotropic damage reads:

$$\boldsymbol{\varepsilon} = \rho\frac{\partial \psi^*}{\partial \sigma} = \frac{1+\nu}{E}[(\mathbf{1}-\boldsymbol{D})^{-1/2}\cdot \sigma^D \cdot (\mathbf{1}-\boldsymbol{D})^{-1/2}]^D + \frac{1-2\nu}{3E}\left[\frac{\langle Tr\sigma\rangle_+}{1-Tr\boldsymbol{D}} + \langle Tr\sigma\rangle_-\right]\mathbf{1} \tag{2}$$

And the strain energy release rate density $\boldsymbol{Y}$ (the thermodynamics force associated with $\boldsymbol{D}$) is $\boldsymbol{Y} = \rho\frac{\partial \psi^*}{\partial \boldsymbol{D}}$, whose expression, quite complex due to the possible rotation of the principal axis, can be found in Lemaitre & Desmorat (2005).

The decoupling between the deviatoric and the volumic part of the stress-strain relation induces only a partial stiffness recovery sufficient for monotonic applications. In compression, damage does not affect the hydrostatic response with the bulk modulus $\tilde{K} = (1 - Tr\boldsymbol{D})k$. In tension the damaged bulk modulus is $K = E/(31-2\nu)$. Note that the shear-bulk coupling is nevertheless represented. The damage $\boldsymbol{D}$ acts (as its trace) on the hydrostatic stresses.

Previous elasticity law can be rewritten as $\tilde{\sigma} = \mathbf{E}:\varepsilon$, with $\mathbf{E}$ the undamaged Hooke tensor. This defines analytically the relationship between the Cauchy stress $\sigma$ and the effective stress $\tilde{\sigma}$, symmetric and independent from the elasticity parameters,

$$\tilde{\sigma} = [(1-\boldsymbol{D})^{-1/2}\cdot \sigma^D \cdot (\mathbf{1}-\boldsymbol{D})^{-1/2}]^D + \frac{1}{3}\left[\frac{\langle Tr\sigma\rangle_+}{1-Tr\boldsymbol{D}} + \langle Tr\sigma\rangle_-\right]\mathbf{1} \tag{3}$$

so that the elasticity laws simply sums up as :

$$\varepsilon = \mathbf{E}^{-1}:\tilde{\sigma} = \frac{1+\nu}{E}\tilde{\sigma} - \frac{\nu}{E}Tr\tilde{\sigma}\mathbf{1} \tag{4}$$

### 2.2. Damage threshold and evolution laws

Damage evolution is linked to the reach or not of a criterion. Depending on the materials, the damage criterion may be expressed, similarly to plasticity, thanks to the stresses (Ortiz 1985, Warnke 1975, Voyiadjis & Abu-Lebdeh 1994), the strains (Mazars 1984, Herrmann & Kestin 1988, Ramtani 1990, de Vree, *et al.* 1995, Geers, *et al.* 2000) or using energy quantities like the damage energy release rate (Marigo 1981, Laborderie, *et al.* 1990). Most of the finite elements codes use displacements based interpolation functions. The most efficient way to express constitutive equations arguing for straightforward numerical integration is to make use of a damage threshold based on strains. For brittle materials like concrete, Mazars's criterion (1984) defining an equivalent strain $\varepsilon_{eq}$ is here used for the anisotropic growth of damage. The equivalent strain $\varepsilon_{eq} = \hat{\varepsilon}$ is built from the positive principal strain $\varepsilon_I$ (the extensions):

$$\hat{\varepsilon} = \sqrt{\langle\varepsilon\rangle_+ : \langle\varepsilon\rangle_+} = \sqrt{\sum \langle\varepsilon_I\rangle_+^2} \tag{5}$$

The damage threshold takes the simple form:

$$f = \hat{\varepsilon} - \kappa(tr\boldsymbol{D}) \le 0 \tag{6}$$

with $f < 0$ corresponding to the elasticity domain and $f$=0 and $\dot{f} = 0$ to damage growth (consistency condition). $\kappa(tr\boldsymbol{D})$ is the consolidation function in the strains space, depending on the trace of the damage tensor. The initial value defines the damage threshold $\kappa_0 = \kappa(0)$.

The advantages are multiple. On the one hand and as just mentioned, the use of equivalent quantities based on strains will make easier the explicit derivation of the subsequent numerical scheme for stress integration. On the other hand, one deals only with one scalar equivalent quantity $\hat{\varepsilon}$ based on strains. The classical drawbacks of spurious mesh dependency when using softening constitutive equations (Bazant 1976, Hillerborg, *et al.* 1976) can simply be numerically avoided by implementing the law within the framework of nonlocal media through the use of a nonlocal weight function $\psi$ (Pijaudier-Cabot & Bazant 1987) or from a gradient equation (Peerlings 1996). It only needs the identification of a new parameter $l_c$, which is the internal length of the nonlocal medium. This length can be linked to the size of the heterogeneities of the material or to the resulting scale effects when dealing with structural analyses. In the following any equivalent strain $\hat{\varepsilon}$ is made nonlocal as

$$\hat{\varepsilon}^{nl}(x) = \frac{1}{V_r(x)}\int_V \hat{\varepsilon}(s)\,\psi(s-x)\,dv \tag{7}$$

with $V_r(x) = \int_V \psi(s-x)\,dv$ and $\psi(s-x) = exp(-2\|s-x\|^2/l_c^2)$

so that the damage criterion becomes $f = \hat{\varepsilon}^{nl} - \kappa(tr\boldsymbol{D}) \le 0$.

Only one damage tensorial variable is used to represent the micro-cracking pattern in concrete. For this model, the non-symmetric tension/compression response is obtained thanks to the anisotropic feature of damage:

− micro-cracks are mainly orthogonal to the loading direction in tension,

− micro-cracks are mainly parallel to the loading direction in compression.

The damage evolution in one direction is guided by the level of extension in this direction. Then the damage can be considered proportional to the positive part of the strain tensor $\langle\varepsilon\rangle_+$ or of its square $\langle\varepsilon\rangle_+^2$ (Ramtani, *et al.* 1992). The power 2 is used next as $Tr\dot{\boldsymbol{D}} \propto Tr\langle\varepsilon\rangle_+^2$ simplifies in $Tr\dot{\boldsymbol{D}} \propto \hat{\varepsilon}^2$, making Mazars strain appears.

Within the thermodynamics framework, the corresponding choice for the non associated potential is:

$$F(\boldsymbol{Y}, \varepsilon) = \boldsymbol{Y}:\langle\varepsilon\rangle_+^2 \tag{8}$$

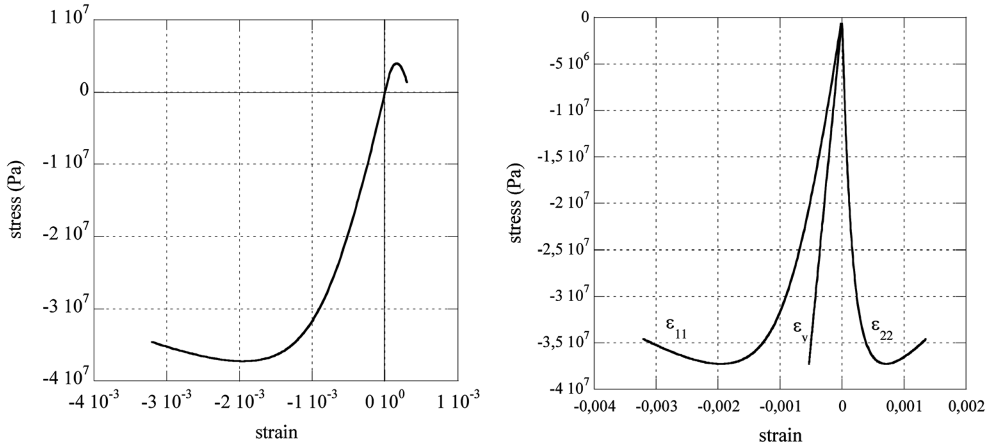
The damage evolution law is obtained by derivation with regard to the strain energy release rate density:

$$\dot{\boldsymbol{D}} = \dot{\lambda}\frac{\partial F}{\partial \boldsymbol{Y}} = \dot{\lambda}\langle\varepsilon\rangle_+^2 \tag{9}$$

The damage multiplier $\dot{\lambda}$ is determined from the consistency condition $f$=0 and $\dot{f} = 0$. It has been demonstrated (Desmorat 2006) that the dissipation due to the damage $\mathcal{D} = \boldsymbol{Y}:\dot{\boldsymbol{D}}$ is a positive quantity for any loading and takes the following form:

$$\mathcal{D} = \boldsymbol{Y}:\dot{\boldsymbol{D}} = \frac{1+\nu}{2E}[\sigma^D \cdot \mathbf{H} \cdot \sigma^D]:\dot{\mathbf{H}} + \frac{1-2\nu}{6E}\frac{\langle Tr\sigma\rangle_+^2}{(1-Tr\boldsymbol{D})^2}Tr\dot{\boldsymbol{D}}, \quad \text{with} \quad \mathbf{H} = (\mathbf{1}-\boldsymbol{D})^{-1/2} \tag{10}$$

The identification of different behaviors is performed by the definition of the consolidation function $\kappa(Tr\boldsymbol{D})$. For concrete, a simple expression has been found which fits the concrete responses in both tension and compression, introducing only two damage parameters $a$ and $A$ in addition to the Young's modulus $E$, the Poisson's ratio $\nu$ and the damage threshold $\kappa_0$,




(a) (b)

Fig. 1 Uniaxial response of the anisotropic damage model. (a) Tension/compression non symmetric feature and (b) volumic behaviour in compression ($\varepsilon_V = Tr\,\varepsilon$)

$$\kappa(Tr\boldsymbol{D}) = a.\tan\left[\frac{Tr\boldsymbol{D}}{aA} + \arctan\left(\frac{\kappa_0}{a}\right)\right] \tag{11}$$

### 2.3. Uniaxial responses

The responses of the model, subject to uniaxial states of stresses are presented in Figs. 1. The material parameters used in the analysis are $E$ = 42 GPa, $n$ = 0.2, $\kappa_0$ = 5 $10^{-5}$, $A$ = 5 $10^3$, $a$ = 2.93 $10^{-4}$.

One can observe that with only one thermodynamic variable (the damage tensor), the model is able to correctly describe the non symmetric uniaxial behaviour of concrete. Due to the decoupling between the hydrostatic part and the volumic part of the model, the volumic strain is not affected in compression. Even if dilatancy is not represented, this is an improvement compared to the loss of bulk modulus obtained with isotropic damage models.

## 3. Numerical implementation

### 3.1. Euler backward scheme for numerical stresses and damage computations

One major advantage of this model is its easy numerical implementation within any finite element code and its corresponding robustness regarding computations at the structural scale. In fact, the constitutive equations can be analytically (straightforwardly or explicitly) calculated, even when using an implicit discretization scheme. No internal iteration, very computer time consuming, are needed at the Gauss point level. Knowing all the variables and stress at time $t_n$ as well as the final state of strain at time $t_{n+1}$, $\varepsilon_{n+1}$, the stress-damage algorithm aims at computing the internal variable (the damage tensor $\boldsymbol{D}_{n+1}$) and the stresses $\sigma_{n+1}$ at time $t_{n+1}$. The principal steps for stress computation are recalled hereafter.

1. Calculate the non local equivalent strain if such a regularisation procedure is adopted (Eq. 7). If not, consider $\varepsilon_{eq}^{nl} = \varepsilon_{eq}$ in the following
2. Calculate the threshold function trial: $f_{trial} = \varepsilon_{eq,n+1}^{nl} - \kappa(tr\boldsymbol{D}_n)$ with any appropriate equivalent strain, local or nonlocal.

If the elasticity criterion is not violated, i.e., $f < 0$, the damage does not evolve ($\boldsymbol{D}_{n+1} = \boldsymbol{D}_n$) the stress increment is obtained thanks to a reversible elastic change of state (go directly to steps 6 and 7). On the contrary, the internal variable has to be corrected, according to the nonlinear constitutive equations and the consolidation function.

3. Discretize the damage evolution as : $\Delta\boldsymbol{D} = \boldsymbol{D}_{n+1} - \boldsymbol{D}_n = \Delta\lambda\langle\varepsilon_{n+1}\rangle_+^2$
4. Taking the trace of the previous expression (This makes the simplification $tr\langle\varepsilon_{n+1}\rangle_+^2 = \hat{\varepsilon}_{n+1}^2$), the damage multiplier increment can be explicitly computed

$$\Delta\lambda = \frac{Tr\boldsymbol{D}_{n+1} - Tr\boldsymbol{D}_n}{\hat{\varepsilon}_{n+1}^2}, \text{ with } Tr\boldsymbol{D}_{n+1} = \kappa^{-1}(\varepsilon_{eq,n+1})$$

5. Update the damage tensor: $\boldsymbol{D}_{n+1} = \boldsymbol{D}_n + \Delta\lambda\langle\varepsilon_{n+1}\rangle_+^2$
6. Calculate the effective stresses: $\tilde{\sigma}_{n+1} = \mathbf{E}:\varepsilon_{n+1}$
7. Determine the Cauchy stresses by inverting Eq (3). Again this is done in an analytical way,

$$\sigma_{n+1} = (\mathbf{1} - \boldsymbol{D}_{n+1})^{1/2} \cdot \tilde{\sigma}_{n+1} \cdot (\mathbf{1} - \boldsymbol{D}_{n+1})^{1/2} - \frac{(\mathbf{1} - \boldsymbol{D}_{n+1}):\tilde{\sigma}_{n+1}}{3 - Tr\boldsymbol{D}_{n+1}}(\mathbf{1} - \boldsymbol{D}_{n+1})$$

$$+ \frac{1}{3}[(1 - Tr\boldsymbol{D}_{n+1})\langle Tr\tilde{\sigma}_{n+1}\rangle_+ + \langle Tr\tilde{\sigma}_{n+1}\rangle_-]\mathbf{1}$$

### 3.2. Rupture control procedure

Describing rupture of structural elements needs to deal with high levels of damage, the ultimate state of degradation corresponding to eigenvalues close to unity. For isotropic damage, respecting the convexity of elasticity law, it is quite clear that the maximum value of the damage variable should not exceed one. Within an anisotropic framework with partial stiffness recovery, such a criterion is no more so simple. The tensorial damage acts by its individual values on the deviatoric part of the behavior and by its trace on the volumic part. It is obvious that one has to consider two different treatments for the damage evolution, depending on which part of the behavior is concerned. Such a treatment must simply ensure that the effective damaged elasticity tensor $\tilde{\mathbf{E}}$ remains positive definite.

A critical value $D_c$ for the principal damages is introduced, allowing defining the numerical transition between a spread micro-cracking pattern and a localized damage up to the occurrence of a macroscopic crack, for which continuum damage mechanics reaches its limit. Transition to nonlinear fracture mechanics must then be considered, for example, by means of nonlinear fracture mechanics or extended finite element formulation. Both handle strong discontinuities in the material (Belytschko & Black 1999, Jirasek 2000). In a first approximation for concrete, $D_c$ is taken equal to 0.99.

Concerning the hydrostatic part of the constitutive equations, the limitations at high level of damage appears clearly when observing the reversible process equations (bulk elasticity from Eq (2)).

$$Tr\varepsilon = \frac{\langle Tr\sigma\rangle_+}{3K(1 - Tr\boldsymbol{D})} + \frac{\langle Tr\sigma\rangle_-}{3K} \tag{12}$$

To keep positive the damaged bulk modulus, it is necessary to limit only in tensile loadings the evolution of the trace of $\boldsymbol{D}$ to $D_c$. In such a way that, when $tr\boldsymbol{D}$ reaches $D_c$, the bulk modulus takes the limiting value of $\tilde{K} = K(1-D_c)$ when $Tr\varepsilon>0$, independently of the values for the damage tensor $\boldsymbol{D}$.

For the deviatoric case, to ensure that the damaged elasticity operator remains positive definite, one only has to impose that the eigenvalues of the second order damage tensor are bounded by 1, or by $D_c$ from a numerical point of view (Lemaitre, *et al.* 2000, Badel 2001). Under these conditions, the general damage law $\dot{\boldsymbol{D}} = \dot{\lambda}\langle\varepsilon\rangle_+^2$ needs an adaptation. If the maximum eigenvalue of damage, $D_I>D_{II}>D_{III}$, reaches its critical value in the direction $n_I$, damage growth in that direction is stopped, defining a first plane of fixed crack in the solid. Damage then goes on by growing in the remaining $(n_{II}, n_{III})$ directions. The damage evolution law is kept unchanged in terms of the (2D) strain tensor in the $(n_{II}, n_{III})$ plane, conserving $D_I = D_c$ along $n_I$. The corresponding (projected) evolution law formally reads:

$$\dot{\boldsymbol{D}} = \dot{\lambda}\Pi_{n_I}\langle\varepsilon\rangle_+^2 \tag{13}$$

with $\Pi_{n_I}$ the projection operator, completely defined in Desmorat, *et al.* (2007). If the loading continues, a second direction $(n_{II})$, for which damage $D_{II}>D_{III}$ reaches its critical value, is detected, defining in the same way, i.e., orthogonally to the two first directions, the third normal $(n_{III})$, and so the eigen base of the damage operator for the cracked medium. Three families of cracks are introduced at the final stage and the fully broken behavior is an elastic one (with different bulk modulus in tension and in compression),

$$\sigma = 2G(1-D_c)\varepsilon^D + K[(1-D_c)\langle Tr\varepsilon\rangle_+ + \langle Tr\varepsilon\rangle_-]\mathbf{1} \tag{14}$$

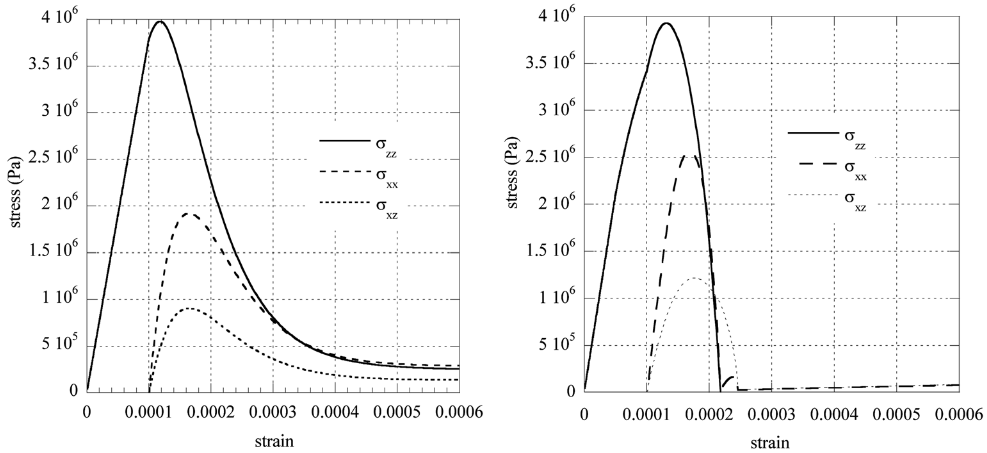
with $G = E/2(1+\nu)$ the shear modulus.

## 4. Biaxial behaviour and equivalent strain

The biaxial behavior of brittle materials like concrete has to be handled with care, due to the complexity of the degradation modes and cracking pattern (Kupfer, *et al.* 1973). The difficulties are twice: the physical representation of complex mechanisms have to be dealt with and the numerical treatment of the constitutive equations has to be robust enough for structural applications. In a first section, the numerical issue is analyzed through a numerical benchmark by comparing an isotropic damage model with an anisotropic one. The biaxial rupture modeling is treated in a second section. The suitable representation of biaxial rupture needs an adequate elasticity limit as well as evolution equations able to deal with confinement effects (van Mier 1984, Feenstra 1993, Lubarda, *et al.* 1996). The second section is devoted to the biaxial response evaluation of the previous model but, original point, using different equivalent strains. Even if damage anisotropy is considered, such an analysis can easily be performed due to the high level of modularity of the constitutive model with for instance a strain-based damage criterion.

### *4.1. Numerical robustness assessment under biaxial loadings*

In a first part, the robustness of the adopted numerical scheme for the stress integration algorithm is assessed using a classical numerical benchmark from the literature (Willam, *et al.* 1987). The




(a) Isotropic Mazars model (b) anisotropic damage model

Fig. 2 Willam's tests results in plane stress state for the (a) isotropic Mazars model and (b) the anisotropic model

main issue of this case-study is to observe the local response of the model in case of principal axis rotation. For anisotropic model, the numerical difficulty in numerical convergence for complex loading path makes this test discriminant. A complete analysis of this numerical benchmark is given in Carol, *et al.* 2(001). The loading is applied in two steps, under displacements control:

– A first uniaxial tension is applied in the direction $z$ till the peak in the stress-strain diagram.

– The second loading consists in a rotation of the strain tensor principal axis by applying a non proportional loading using biaxial-tension and shear $(\Delta\varepsilon_{xz})$. The applied strain increments are as follows : $\Delta\varepsilon_{zz} = 0.5\Delta\varepsilon_{xz}$, $\Delta\varepsilon_{xx} = 0.75\Delta\varepsilon_{xz}$ .

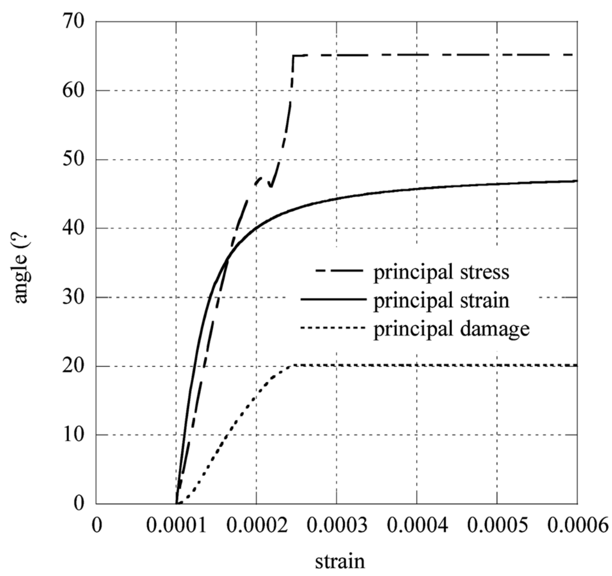



Fig. 3 Rotation of the principal direction for the Willam's test in case of anisotropic modelling

The original isotropic Mazars model as well as the anisotropic damage model have been studied throughout this numerical study.

Fig. 2 shows the response for the three stresses components in function of the axial strain $\varepsilon_{zz}$ for the two models under investigation.

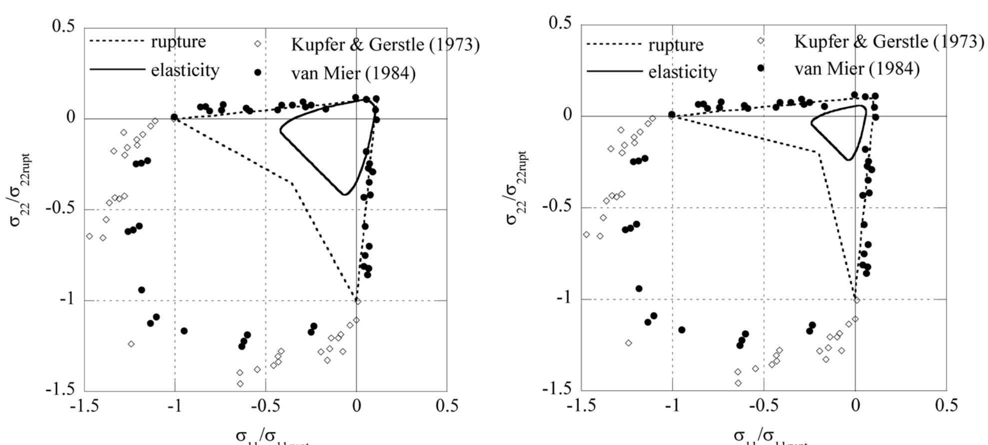
As one can see, the dissipated energy up to rupture is bounded for the anisotropic model and not for the isotropic one. The difference between an isotropic and an anisotropic model is more obvious when looking at the difference of major principal direction angles for the stresses tensor and the strains tensor. For an isotropic model, the rotation of the stresses principal directions follows the strains one. For an anisotropic model, these two sets of coordinates differ. As the rotations under an isotropic framework are equal, fig. 3 only shows the response concerning the major principal direction angle for the anisotropic damage model. The rotation of the principal axis of the damage tensor is also plotted, function of the applied strain in the *z* direction.

The fast evolution of the stresses principal direction is due to the rapid softening of the 2 normal stresses regarding a softer evolution of the shear stress in the post-peak regime. The damage principle direction rotates continuously till its first maximum principal value $D_I$ reaches $D_c$, for a vertical strain of 0.24%. For a biaxial state of stresses, the 3 principal directions are completely defined and kept unchanged after this point. The three stresses components are closed to 0, and the broken behaviour is then considered to the end. Similarly to a class of anisotropic damage model (Badel, *et al.* 2007), the broken asymptotic behaviour with fixed principal directions for damage is here recovered. The evolution of the stresses principal axis goes on growing, with an almost 'plateau' shape.

The numerical simulation points out the robustness of the discretization scheme and convergence for a complex biaxial loading.

### 4.2. Biaxial responses for an isotropic model and for the initial anisotropic model

The constitutive models based on continuum damage mechanics, isotropic or anisotropic, exhibit a




(a) Isotropic Mazars model (b) Initial anisotropic model

Fig. 4 Elasticity domain and ultimate state at rupture for (a) the isotropic model (Mazars 1984) and (b) the initial anisotropic model using the Mazars equivalent strain high level of brittleness in biaxial compression. If the state of stress at rupture is plotted in the $(\sigma_{11}, \sigma_{22})$ plane, one can observe a decrease of the load bearing capacity in biaxial compression (with regards to the uniaxial response in compression, $\sigma_{rupt}$) while the experimental results show an increase of the stress at rupture of about 20% (Kupfer & Gerstle 1973, van Mier 1984). In Fig. 4(a), one shows the elasticity limit as well as the rupture envelop, using isotropic Mazars damage model (and of course Mazars equivalent strain). Kupfer (1973) and van Mier (1984) experimental data are drawn in the figure.

Fig. 4(b) is due to the initial anisotropic damage model with Mazars equivalent strain in the damage criterion. The brittleness increase is even more clearly emphasized. The effects of damage on the non convexity of the rupture limit are even more pronounced for this anisotropic model, by localizing the micro-cracks in a unique plane, orthogonal to the loading plane. The lack of convexity in biaxial compression of the initial elasticity domain (the flat end with Mazars's criterion) is the major source for the bad representation of biaxial responses. A possible remedy for compressive loadings is of course the consideration of both plasticity and damage mechanisms but the corresponding models becomes quite complex (Meschke & Lackner 1998, Ragueneau *et al.* 2000, Jason, *et al.* 2006). A second remedy, much simpler in terms of numerical efforts, consists in keeping the elasticity coupled with damage but in increasing the non-symmetry of the elasticity domain by adding new invariants in the expression of the equivalent strain. This is the choice made in the present work.

### 4.3. Modification of the equivalent strain

Different solutions may be adopted to improve the biaxial responses of the models subject to complex states of stress. By only changing the definition and expression of the elasticity domain, the numerical integration is kept unchanged as well as the nonlocal formulation allowing for proceeding structural case study without mesh dependency. Different formulations can be adopted for concrete, based on the original expression of the Mazars equivalent strain $\hat{\varepsilon} = \sqrt{\langle\varepsilon\rangle_+ : \langle\varepsilon\rangle_+}$ completed by terms function of the strain invariants:

$I_1 = I_1(\varepsilon) = Tr(\varepsilon)$ and $.J_2 = J_2(\varepsilon) = \frac{1}{2}\left[\varepsilon : \varepsilon - \frac{1}{3}I_1^2\right]$

- Mazars-Drucker-Prager: adding of the trace of the strain tensor. A complementary material parameter $k$ has to be identified. Although it greatly improves the elasticity domain in bi-compression, the main disadvantage of this expression is to modify in the same way the tensile response making necessary a complete reidentification of all the material parameters.

$$\varepsilon_{eq} = \hat{\varepsilon} + kI_1 \tag{15}$$

- Modified Mazars-Drucker-Prager: adding of the trace of the negative part of the strain tensor, without modifying the response in tension. This non-convex surface may generate, depending on the material parameters choice, instability of the mechanical response.

$$\varepsilon_{eq} = \hat{\varepsilon} + k\langle I_1\rangle_- \tag{16}$$

- Mazars-Mises-Drucker-Prager: adding of the trace of the strain tensor and of the second invariant of the deviatoric part of the strain tensor. This equivalent strain defines a convex elasticity domain making easier the constitutive equation parameters identification. The $1/\sqrt{2}$

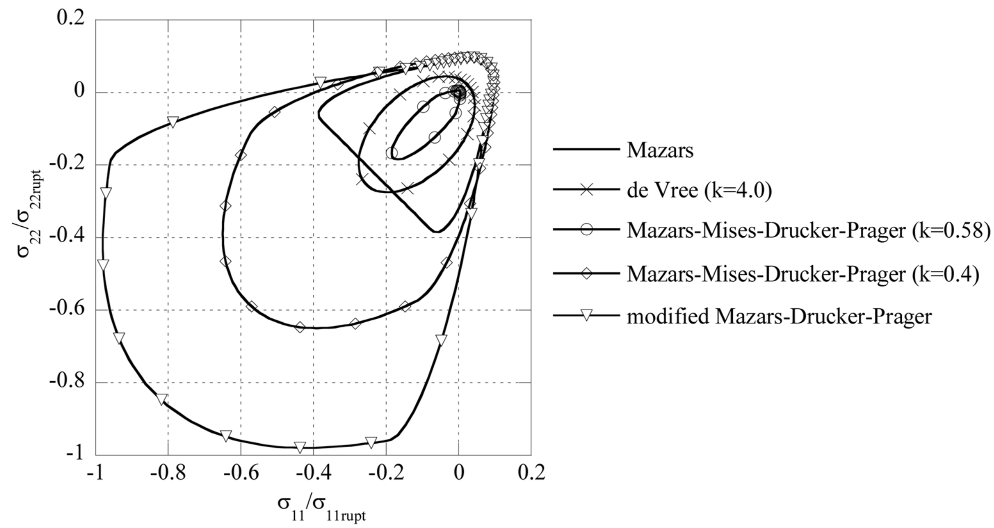



Fig. 5 Elasticity domains comparisons for the different equivalent strains. The $k$ parameter has been adjusted to obtain for each model the same responses in tension and in compression as in fig. 1

value can be replaced by a material parameter but it suits well concrete materials.

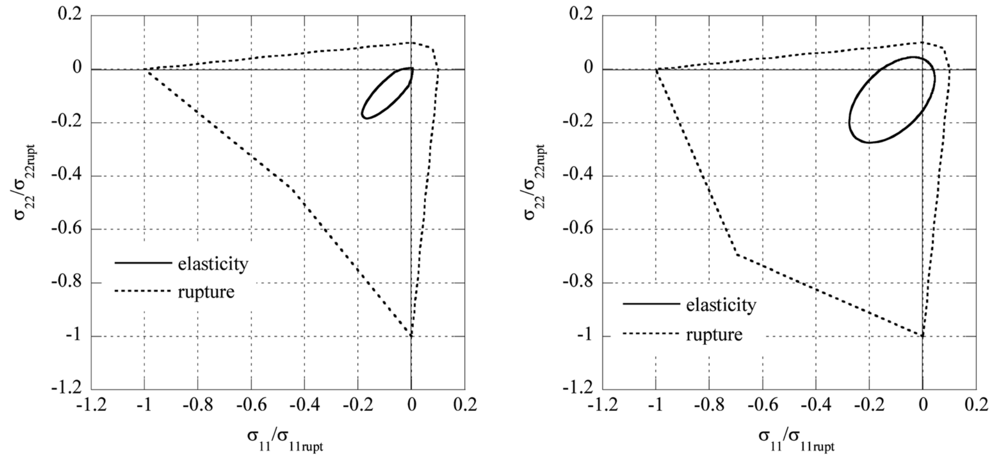
$$\varepsilon_{eq} = \hat{\varepsilon} + kI_1 + \frac{1}{\sqrt{2}}\sqrt{J_2} \tag{17}$$

- De Vree criterion: a non-centered von Mises criterion in the strain space (de Vree, *et al.* 1995). The ellipsoidal shape, far from the previous one, allows for a direct control of the non-symmetric behavior of concrete through the introduction of the material parameter $k$.

$$\varepsilon_{eq} = \frac{k-1}{2k(1-2\nu)}I_1 + \frac{1}{2k}\sqrt{\frac{(k-1)^2}{(1-2\nu)^2}I_1^2 + \frac{12k}{(1+\nu)^2}J_2} \tag{18}$$

The elasticity domains using all these equivalent strains are plotted in Fig. 5. The constitutive relations parameters for the anisotropic damage model have been identified for each surface, so as to feet the same behavior as presented in fig. 1. For comparisons, the elasticity domains and rupture envelops are plotted in Fig. 6 for the de Vree equivalent strain and the Mazars-Mises-Drucker-Prager one. For these two examples, the modification of the equivalent strain allows for a better simulation of the biaxial behavior of concrete. The de Vree surface seems to be more appropriate, without increasing in a too much important manner the non-symmetric threshold for elasticity.

Such a difference in the biaxial behavior is more relevant when looking at the shear response of the different models. The isotropic Mazars model in simple shear is compared to the anisotropic one using the previous different expressions for the equivalent strain and elasticity surface. The shear behaviour of concrete, due to aggregate interlocks, roughness and frictional sliding along the crack surfaces should be more resistant than the response in pure tension. This point is illustrated in Fig. 7, especially for the de Vree surface in terms of bearing capacity and for the isotropic Mazars model in terms of ductility.




(a) (b)

Fig. 6 Elasticity domain and ultimate state at rupture for (a) the anisotropic damage model using Mazars-Mises-Drucker-Prager equivalent strain and (b) the de Vree Criterion

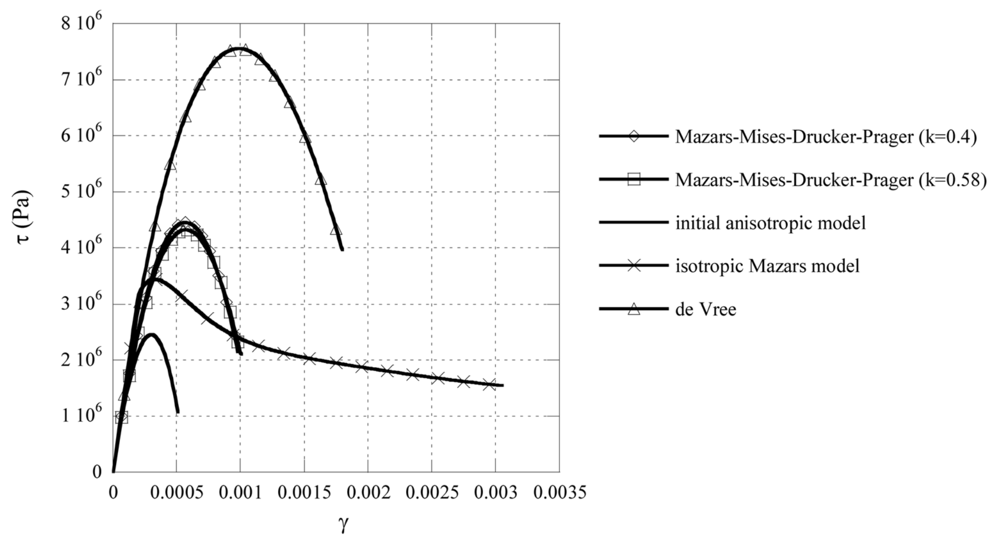



Fig. 7 Shear response for the isotropic and the anisotropic damage models using the different expressions for the equivalent strains

## 5. Nooru-Mohamed's structural case-study

The classical experiment of Nooru-Hohamed (1992) is used to analyse, at the structural level, the effects of the changes in the expression of the equivalent strain. Due to the interaction between tension and shear following a non proportional path, this test is relevant for quantifying the effects of the different equivalent strains in our modeling.

The specimen geometry and the experimental testing set up are shown in Fig. 8 It is a symmetric 200 mm · 200 mm mortar square with two notches, 30 mm long and 5 mm thick. The case study is here carried out for a maximum shear load $F_{Max}$ = 10 kN exhibiting mixed mode fracture. The 3D

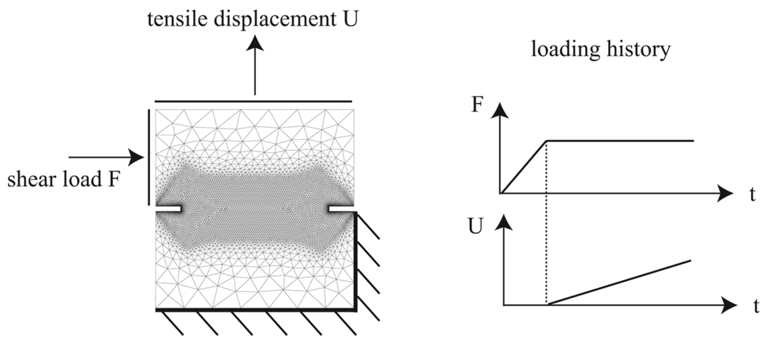



Fig. 8 Nooru-Mohamed test. Mesh, boundary conditions and loading history

Finite Element discretization of the specimen is made by the use of four node tetrahedron elements with one integration point. In order to perform the computations in 3D at reasonable cost, a FE mesh with a 5 mm width is used when the real width of the specimen is 50 mm. The mesh, the boundary conditions as well as the loading specifications are presented in Fig. 8.

The model parameters used for the simulation are those of section 2 for concrete: $E$ = 42000 MPa, $\nu$= 0.2, $\kappa_0$= 5.10$^{-5}$, $A$ = 5.10$^3$, $a$ = 2.9310$^{-4}$.

In order to avoid any spurious mesh dependency due to strain localisation, nonlocal computations are performed on the mesh of Fig. 8. The characteristic length is set to $l_c$ = 2 mm for all the analyses. The convergence of the computations has been numerically shown using three types of meshes in a previous work (Desmorat, *et al.* 2007). The results are given in the following using a medium mesh. Two computations are here performed using the isotropic Mazars damage model and the anisotropic damage model including the different equivalent strains (original Mazars, de Vree and Mazars-Mises-Drucker-Prager). The same characteristic length has been used for the different

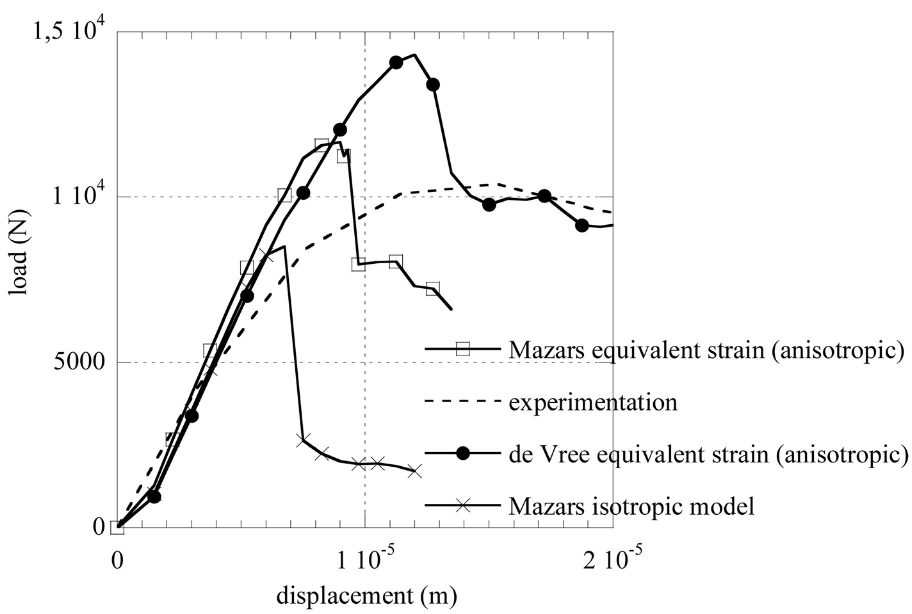



Fig. 9 Nooru-Mohamed test. Comparisons of different types of damage threshold during the loading

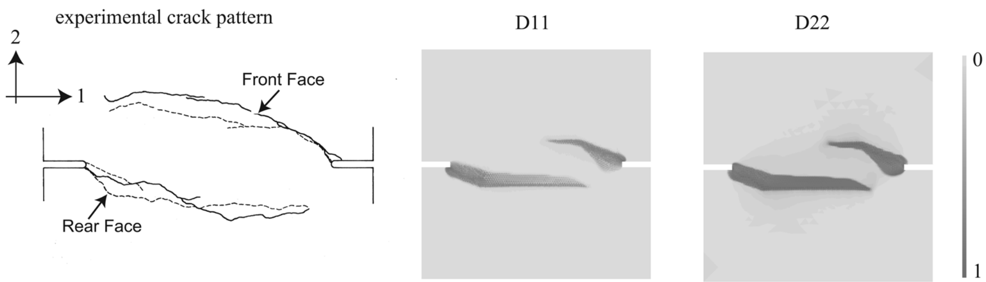



Fig. 10 Nooru-Mohamed test. Experimental crack pattern. Damage fields in the direction 1 (horizontal) and 2 (vertical)

computations. The different responses are plotted in Fig. 9 in the tensile load – vertical displacement diagram, corresponding to the second step of the loading history. For biaxial state of stresses, damage induced anisotropy helps to capture rupture in a better way than an isotropic damage description. Due to the 'mixing' of the cracking mode in a single scalar variable for the isotropic model, the rupture occurs prematurely. For comparison purposes, the characteristic length has been chosen equal for the isotropic model and for the anisotropic one. Regarding energy dissipation, the link between isotropic and induced anisotropic damage model may explain some part of the major difference between the two responses. Using anisotropic model, three responses are plotted in Fig. 9, with original anisotropic damage model, i.e., the model with the damage threshold based on nonlocal Mazars equivalent strain only, and with the nonlocal de Vree equivalent strain. The response using the Mazars-Mises-Drucker-Prager nonlocal equivalent strain is not reported here: the relatively small initial elasticity domain generates rupture and lack of convergence during the step1-step2 transition. The ductility as well as the peak displacement is better simulated using the de Vree equivalent strain. Although it allows for a better response modeling in the biaxial regime, the de Vree criterion overestimates the shear response. This point may explain the overload registered in the numerical computations in Fig. 9.

The comparisons between the experimental crack pattern and the anisotropic $D_{11}$ and $D_{22}$ damage fields are plotted in the Fig. 10 (note that direction 1 is horizontal and direction 2 is vertical). Whatever the criterion, the damage fields obtained are quite close so that only the results obtained with the original Mazars equivalent strain are plotted in fig. 10. One can observe that the crack path is recovered by the numerical computations. A non-symmetric cracking pattern is obtained. This fact is due to the use of an explicit rupture control procedure. An implicit scheme should be adopted in future developments to circumvent such a drawback.

## 6. Conclusions

Dealing with multiaxial behavior of brittle materials like concrete needs to account for micro-cracks orientations. A 3D robust model introducing damage induced anisotropy is presented in this paper, as well as its stresses computation numerical scheme. Different equivalent strain-based damage criterion are implemented in a finite element code and compared on a structural case study. Important feature, even if Euler backward scheme is used, no iterations are needed at the Gauss point level. The full anisotropic damage model only needs the consideration of a single thermodynamics but tensorial damage variable and the identification of 6 material parameters (2 for elasticity, one as damage threshold, 2 for damage growth and a new parameter $k$ important for both the shear and the biaxial responses) to handle the non-symmetric response of concrete in tension and in compression. Benchmarking the models responses with the different equivalent strains under biaxial compression emphasizes the lack in ductility encountered by models based on continuum damage mechanics with no plasticity, whatever the damage kinematics, scalar or tensorial. The modifications of the damage threshold and the definition of the new equivalent strains improve the multiaxial response. The elasticity domain, defined thanks to the different equivalent strains and the high level of modularity of the anisotropic damage models here studied, allows for 'parametric' analyses of different yield surfaces and for quantifying their role on failure envelops. The equivalent strain, made modular by the addition of strain tensor invariants, is modified to account for a better dissymmetry in the elastic response of the material in tension and in bi-compression. de Vree equivalent strain, based on a modified non-centered von Mises criterion proves its efficiency to deal with bi-compressive states of stresses. When applied to a structural case study of a concrete sample subject to both shear and tension, this equivalent strain, coupled with the anisotropic damage model predicts an overestimation of the specimen load bearing capacity, due to the local shear ductility increase.

Some intermediate solutions will be retained in future works to improve the numerical responses of induced anisotropic damage models. For example, the permanent strains, as well as the confinement effects induced by dilatancy will help the model to catch the ductility increase in confined cases without penalizing the shear response. This can be done by means of a coupling with plasticity but also from the direct definition of permanent strains due to damage (Hermann & Kestin 1988). But in any case, the definition of an equivalent strain is the key-feature for 3D modeling.

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