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Probabilistic models for curvature ductility and moment redistribution of RC beams

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Abstract. It is generally accepted that, in the interest of safety, it is essential to provide a minimum level of flexural ductility, which will allow energy dissipation and moment redistribution as required. If one wishes to be uniformly conservative across all of the design variables, curvature ductility and moment redistribution factor should be calculated using a probabilistic method, as is the case for other design parameters in reinforced concrete mechanics. In this study, simple expressions are derived for the evaluation of curvature ductility and moment redistribution factor, based on the concept of demand and capacity rotation. Probabilistic models are then derived for both the curvature ductility and the moment redistribution factor, by means of central limit theorem and through taking advantage of the specific behaviour of moment redistribution factor as a function of curvature ductility and plastic hinge length. The Monte Carlo Simulation (MCS) method is used to check and verify the results of the proposed method. Although some minor simplifications are made in the proposed method, there is a very good agreement between the MCS and the proposed method. The proposed method could be used in any future probabilistic evaluation of curvature ductility and moment redistribution factors.

Keywords: RC beams; curvature ductility; moment redistribution; reliability; monte carlo simulation

1. Introduction

Ductility is the ability of a component or an assembly of components to deform beyond the elastic limit. It is expressed as the ratio between the maximum value of a deformation quantity and the same quantity at the yield limit state (Elnashai and Di Sarno 2008). Ductility is a desirable structural property, because it allows stress redistribution and provides warning of impending failure. From a safety point of view, ductility is as critical as strength (Wilson 2009, Kwan and Ho 2010). Reinforced concrete (RC) beams are under-reinforced by design, so that failure is initiated by the yielding of the steel reinforcement, followed (only after considerable deformation and at no substantial loss of load carrying capacity) by concrete crushing and ultimate failure. For beams in seismic-resistant structures (which are designed to be subject to greater flexural ductility demands), tougher requirements for reinforcement detailing are generally imposed, such as the provision of confining reinforcement. However, even for beams which are not expected to resist impact or

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seismic loads, it is generally considered essential to provide a minimum level of flexural ductility (Ho *et al.* 2004).

RC beams are loaded with different patterns of live load. Therefore, for any load combination of these live-load patterns, a certain critical section along the beam reaches its ultimate strength, while other sections have additional capacity at which the load could be redistributed. In elastic analysis, this reserve capacity is not utilised. However, a full inelastic analysis based on hinge formation can take advantage of this reserve capacity. The most common way of dealing with this is to perform the analysis elastically, but to use a moment redistribution factor to consider the redistribution. The amount of moment redistribution depends on the ductility of the inelastic regions, the geometry of the beams and the loading pattern.

The strength and ductility capacities of RC members depend on various geometric and material properties, most of which are random in nature. Therefore, there is always uncertainty concerning the strength and ductility of RC members. Realistic descriptions of strength and deformation require probabilistic models and the implementation of a reliability-based analysis. Compared to the number of reliability studies conducted on strength limit states, fewer studies have been carried out on ductility limit states, with most of them relying on the MCS method for the analysis (Trezos 1997, Kappos *et al.* 1999, Lu *et al.* 2005).

In this study, using basic simplifications, a closed-form expression for curvature ductility and moment redistribution is derived, based on the method developed by Silva and Ibell (2008). A probabilistic analysis is then performed and a closed-form solution is proposed for the statistical model of curvature ductility and moment redistribution factor. The MCS method is also used to confirm and verify the output of the proposed method, which relies on simple basic equations..

2. Mechanical concept

The provision of minimum ductility is an important requirement imposed by the design codes. This minimum ductility in RC members relates to curvature ductility. However, because of difficulties in quantifying the curvature ductility in RC members, design codes apply other kinds of controls as well, such as imposing limits on rebar percentage or on the tensile strain at the furthest steel rebar. On the other hand, the moment redistribution in continuous RC beams is one of the simplest applications of member ductility in the design procedure.

For conventional RC beams with normal reinforcement, the slope of the post-yield part of the steel strain-stress and moment-curvature curves can usually be neglected. This always produces results that are more conservative. In a similar manner, the effect of compressive rebar is also neglected in the current study for the sake of simplicity, despite it having a positive effect on curvature ductility. Nevertheless, the proposed method could be extended to include compression rebar in future studies.

2.1 Curvature ductility

Derivation of curvature ductility requires a nonlinear section analysis. However, by the basic simplifications made in this study, and by using the equivalent stress block for concrete, it can be derived as a closed form. Fig. 1 shows the mechanical principles that are used in the design of RC beams. The width of the equivalent stress block is given by the product $k_1 k_3 f_3'$. The factor k_2 represents the stress block depth factor.

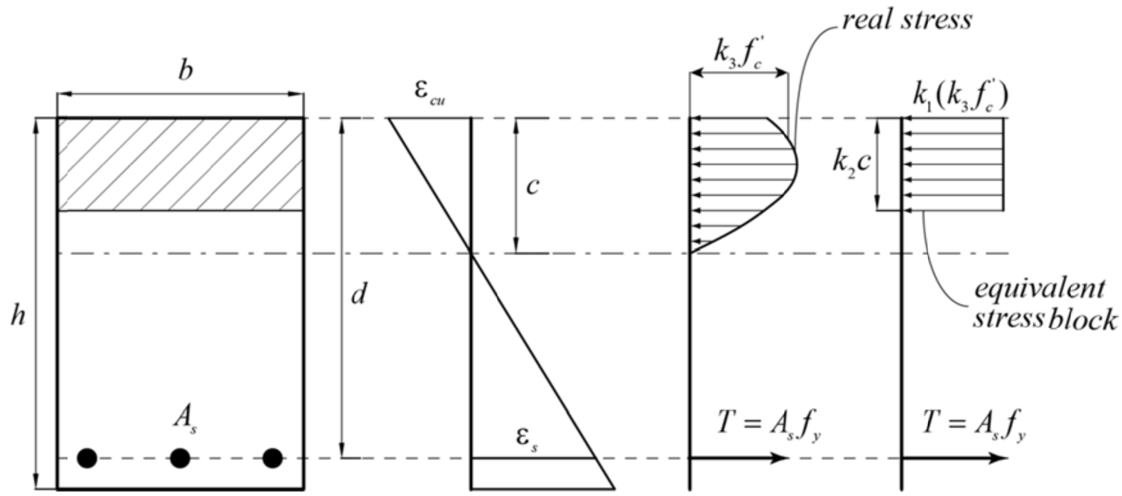


Fig. 1 Basic assumptions in the mechanical model for RC beam section

The ultimate deformation capacity is expressed through the ultimate curvature of the section. The ultimate curvature is the state at which either the specific ultimate compressive strain in the concrete or the specific ultimate strength of extreme tensile rebar is reached. The ultimate compressive strain of unconfined concrete is relatively low and the rebar steel, even for high-strength concrete, has adequate ductility prior to rupture. Therefore, for unconfined concrete, which is the assumed case for under-reinforced beams, reaching the ultimate compressive strain in concrete governs the ultimate curvature. In this study, it is assumed that the ultimate curvature is governed by the crushing of the extreme fibre of the RC beam section.

The curvature ductility capacity for the critical section is derived using the basic mechanics of reinforced concrete. As shown in Eq. (1), the curvature ductility is defined as the ratio of ultimate to yield curvature.

$$\mu_\varphi = \frac{\varphi_u}{\varphi_y} \tag{1}$$

The ultimate curvature is defined as the gradient of strain over the section height. Using geometry, compatibility and equilibrium, Eq. (2) can be easily obtained from Fig. 1.

$$\varphi_u = k_1 k_2 k_3 \frac{f_c'}{f_y} \frac{1}{\rho} \varepsilon_{cu} d \tag{2}$$

The curvature at first yield of tension steel (φ_y) may be found by Eq. (3).

$$\varphi_y = \frac{f_y / E_s}{d(1-k)} \tag{3}$$

In the case of yielding steel, the stress in the extreme compressive fibre of concrete could be appreciably lower than the cylinder strength, $f_c^{'}$. The stress-strain curve for concrete is approximately linear up to $0.70 f_c^{'}$ (Park and Paulay 1975). Therefore, by using elastic theory and assuming that the concrete stress does not exceed this value when the extreme steel yields, the neutral axis parameter at yield is calculated as shown in Eq. (4).

$$k = \rho n\left[\sqrt{1+\frac{2}{\rho n}}-1\right] \tag{4}$$

In Eq. (4), ρ is the tensile rebar percentage and n is the ratio of modulus of elasticity of steel to concrete. By substituting Eqs. (2) to (4) in Eq. (1), the final expression for curvature ductility in a singly-reinforced rectangular beam section can then be derived, as shown in Eq. (5).

$$\mu_\varphi = k_1 k_2 k_3 \frac{f_c^{'}}{f_y^2} E_s \varepsilon_{cu} \frac{1-\rho n\left(\sqrt{1+\frac{2}{\rho n}}-1\right)}{\rho} \tag{5}$$

In Eq. (5), factor $k_1 k_2 k_3$ represents the equivalent stress block parameters, while the $\frac{f_c^{'}}{f_y^2} E_s \varepsilon_{cu}$ factor relates to material properties. The last multiplier represents the cross-sectional dimensions. Eq. (5) represents the capacity of a rectangular section for curvature ductility, which can be thought of as a kind of deflection capacity. By rearranging the parameters appearing in Eq. (5) and using α_1 and β_1 instead of k_1k_3 and k_2 as stress block parameters, Eq. (6) results.

$$\mu_\varphi = \alpha_1 \beta_1 \frac{f_c^{'}}{f_y^2} E_s \varepsilon_{cu} \frac{1}{\rho} g \tag{6}$$

In Eq. (6), the parameter g is defined as $g = 1-\rho n\left(\sqrt{1+\frac{2}{\rho n}}-1\right)$. By this simplification, the curvature ductility is written in the form of a product of different variables that are related to the geometry and material specifications.

2.2 Moment redistribution factor

Possibly the simplest application of member ductility is the ability of a RC flexural member to redistribute moment (Oehlers *et al.* 2010). The basic idea for moment redistribution in continuous RC beams is that the demand rotation required for the development of plastic hinges at the ends and middle of spans should be less than the rotational capacity of the plastic hinge or hinges that yield first. As shown in Fig. 2, a beam that is fixed at both ends, and which can approximately represent an interior span of a multi-span beam, is considered.

The rotational capacity, or ductility, in members can easily be transformed to section curvature ductility, using the concept of plastic hinge length. Equivalent plastic hinge length is used to find the rotational capacity of the plastic region along a beam. It is assumed that the RC beam has a constant flexural stiffness, EI, along its length, and that plastic hinges are first formed at the ends of the beam. The end hinges should show adequate ductility and should deform sufficiently to allow the formation of another hinge at the middle of the beam span.

The demand ductility (rotational or curvature) depends on the geometry of the RC beam, the type of loading and the plastic hinge length in critical regions. Referring to Fig. 2 and using the moment-area method, the demand rotation for the formation of plastic hinges at the ends and the middle of the beam can be calculated as shown in Eq. (7).

$$\theta_{demand} = \frac{l}{2EI}\left(\frac{\omega_u l^2}{12} - M_u\right) \tag{7}$$

Using the concept of equivalent plastic hinge length, the rotational capacity of end hinges can also be calculated as shown in Eq. (8), (Park and Paulay 1975).

$$\theta_{capacity} = (\varphi_u - \varphi_y) l_p \tag{8}$$

In Eq. (8), φ_y and φ_u are the yield and ultimate curvature at the end sections of the beam, while l_p is the equivalent length of the plastic hinge. Generally, due to the high complexity and difficulty involved, the behaviour of plastic hinges in RC members is investigated experimentally (Zhao *et al.* 2012). As shown in Table 1, in this study, one of these empirical expressions for plastic hinge length is used. The relationship between the parameters of ultimate uniform load, ω_u, moment redistribution factor, β, ultimate moment at beam ends, M_u, and elastic moment, M_e, can be written simply as in Eq. (9).

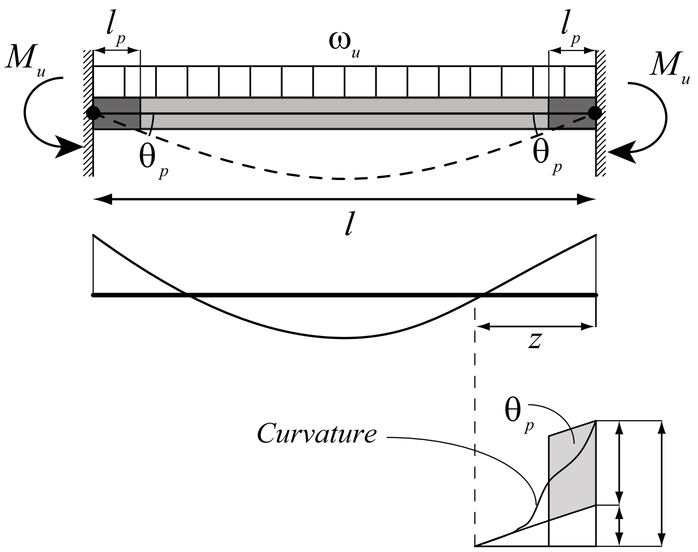


Fig. 2 Typical RC beam geometry

Table 1 Summary of statistical models of random variables

Variable		Bias[1]/Mean	COV[2]/Std[3]	PDF[4]	Reference
Dimension	b	1.00	0.04	Normal	Szerszen and Nowak (2003)
	d	1.00	0.04	Normal	
	A_s	1.00	0.015	Normal	
Concrete	$f_c^{'}$	Nominal+7.5 MPa	6.0 MPa	Lognormal	Attard and Stewart (1998)
	E_c	$1.01\left(4370.3 f_{cm}^{'\ 0.5164}\right)$	0.15	Normal	
	ε_{cu}	$\frac{4.11 f_{cm}^{'}}{E_c \sqrt[4]{f_{cm}^{'}}} \times 2.8133\left(f_{cm}^{'}\right)^{-0.2093}$	0.19	Normal	
	α_1	$1.2932\left(f_c^{'}\right)^{-.0998}$	0.09	Normal	
	β_1	$1.0948\left(f_c^{'}\right)^{-0.091}$	0.03	Normal	
Rebar steel	E_s	1.005	0.033	Lognormal	Lu and Gu (2004)
	f_y	489 MPa	0.068	Beta	Bournonville *et al.* (2004)
Plastic hinge	l_p	$0.077z + 8.16d_b$	0.198	Normal	Lu and Gu (2004)

1 - Mean/Nominal
2 - Coefficient of Variation
3 - Standard Deviation
4 - Probability Density Function

$$M_u = \frac{\omega_u l^2}{12}(1-\beta) = M_e(1-\beta) \tag{9}$$

Equating Eqs. (7) and (8) and substituting Eq. (9) results in Eq. (10) for the demand curvature ductility at the critical end sections.

$$\mu_\varphi = 1 + \frac{1}{2}\left(\frac{\beta}{1-\beta}\right)\left(\frac{l}{l_p}\right) \tag{10}$$

The curvature ductility demand in Eq. (10) should be equal to the curvature ductility capacity obtained in Eq. (6). Rearranging Eq. (10), the allowable β for a fixed-end beam with singly reinforced rectangular section can be obtained as per Eq. (11).

$$\beta = 1 - \frac{1}{1 + 2\left(\frac{l_p}{l}\right)\left(\mu_\varphi - 1\right)} \tag{11}$$

Eqs. (6) and (11) are used as basic equations to calculate curvature ductility and moment redistribution factor in this study.

3. Statistical models

The important parameters for estimating the curvature ductility and moment redistribution factor are shown in Eqs. (6) and (11). The curvature ductility is a function of many other parameters that are related to dimensions and materials. In this section, all random variables are reviewed and the nominal, mean and standard deviations, as well as best-fit probability density function for each variable are selected from the available literature.

Uncertainty in the concrete equivalent stress block and the ultimate strain of concrete require a full statistical analysis of the stress-strain curve of the concrete material. The statistical models for the equivalent stress block parameters of concrete are taken from the study by Attard and Stewart (1998). In their probabilistic analysis of concrete stress blocks, they used the lognormal distribution as the probability density function for the concrete compressive strength. Statistical models for other main random variables are derived from current literature. A summary of the probabilistic models for all basic random variables is presented in Table 1. In this table, f_c' represents the specified or nominal concrete compressive strength, and f_{cm}' shows the mean concrete compressive strength of concrete.

4. Probabilistic analysis

The curvature ductility of RC beam sections and the moment redistribution factor can be predicted by means of a structural analysis based on the material properties and member geometry. Eqs. (6) and (11) represent examples of these predictions. The main variables, such as sectional dimensions or material properties, are random in nature, and as such the member's behavioural parameters (like curvature ductility and moment redistribution factor) are probabilistic.

4.1 Curvature ductility

According to Eq. (6), the curvature ductility is a product of random variables which, based on central limit theorem (Benjamin and Cornell 1975), can be modelled by a lognormal distribution. In order to find its mean and standard deviation, the mean and standard deviation of parameter g should first be calculated. This parameter is a function of ρn which, in turn, is a function of the sectional dimensions and modulus of elasticity of steel and concrete. We consider ρn to be a random variable that is again a product of other random variables and thus follows lognormal distribution. In order to simplify the calculation, the g function in Eq. (6) is approximated as shown in Eq. (13).

$$g = 1 - \rho n\left(\sqrt{1+\frac{2}{\rho n}} - 1\right) = 1 + \rho n - \sqrt{2\rho n} \tag{13}$$

The magnitude of ρn is relatively small, so the number 1 under the square root could be neglected. In order to find the mean and the standard deviation of g, the mean and the standard deviation of this function should first be evaluated. Assuming independency amongst these random variables, and by taking advantage of central limit theorem, the parameter *ρn* is expected to follow lognormal distribution. Its nominal value, bias factor and coefficient of variation are calculated as shown in Eqs. (14) to (17).

$$t_n = (\rho n)_n = \frac{A_{sn} E_{sn}}{b_n d_n E_{cn}} \tag{14}$$

$$\lambda_t = \frac{\lambda_{A_S} \lambda_{E_S}}{\lambda_b \lambda_d \lambda_{E_C}} \tag{15}$$

$$V_t = \sqrt{V_{A_S}^2 + V_{E_S}^2 + V_b^2 + V_d^2 + V_{E_C}^2} \tag{16}$$

$$\sigma_{\ln t}^2 = \ln(1+V_t^2) \quad \& \quad \mu_{\ln t} = \ln(\mu_{\ln t}) - \frac{1}{2}\ln(1+V_t^2) \tag{17}$$

In Eqs. (14) to (17), subscript *n* denotes the nominal value of the random variables. Parameters *V* and *λ* represent the coefficient of variation and bias factor, respectively. The probability distribution function of variable *g* is not of importance, and only its mean and standard deviation are required in order to find the mean and coefficient of variation of curvature ductility. Using Eq. (13) and taking advantage of specific properties of lognormal distribution, the expectation and the variance of the function *g* can be calculated as shown in Eqs. (18) to (23).

$$g_n = 1 + t_n - \sqrt{2t_n} \tag{18}$$

$$E(g) = 1 + E(t) - E(\sqrt{2t}) \tag{19}$$

$$Var(g) = Var(t) + 2Var(\sqrt{t}) - 2\sqrt{2}Cov(t,\sqrt{t}) \tag{20}$$

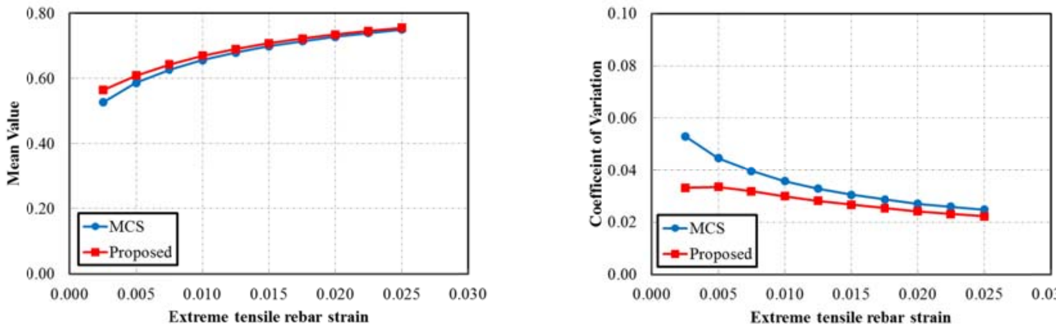
$$Var(t) = E(t^2) - E^2(t) \quad \& \quad Var(\sqrt{t}) = E(t) - E^2(\sqrt{t}) \tag{21}$$

$$Cov(t,\sqrt{t}) = E(t\sqrt{t}) - E(t)E(\sqrt{t}) \tag{22}$$

$$E(t^\alpha) = \exp(\alpha\mu_{\ln t} + \frac{1}{2}\alpha^2\sigma_{\ln t}^2) \tag{23}$$

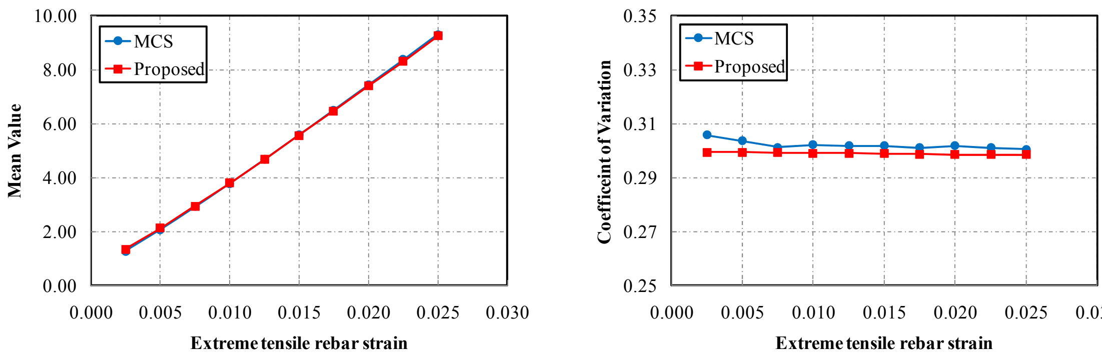
Using the above equations, the bias factor and the coefficient of variation of function *g* are calculated as shown in Eqs. (24) to (26).

$$\mu_g = E(g) = 1 + e^{\mu_{\ln t} + \sigma_{\ln t}^2/2} - \sqrt{2}e^{\mu_{\ln t}/2 + \sigma_{\ln t}^2/8} = 1 + \mu_t - \frac{\sqrt{2\mu_t}}{(1+V_t^2)^8} \tag{24}$$



(a) Mean value (b) Coefficient of variation

Fig. 3 Comparison of the proposed and the MCS methods for calculating statistical properties of variable g



(a) Mean value (b) Coefficient of variation

Fig. 4 Comparison of proposed and MCS methods for statistical properties of curvature ductility

$$\sigma_g = \sqrt{Var(g)} = \sqrt{e^{2\mu_{\ln t}+\sigma^2_{\ln t}}(e^{\sigma^2_{\ln t}}-1) + e^{\mu_{\ln t}+\sigma^2_{\ln t}/4}(e^{\sigma^2_{\ln t}/4}-1) - 2\sqrt{2}e^{\frac{3}{2}\mu_{\ln t}+\frac{5}{8}\sigma^2_{\ln t}}(e^{\sigma^2_{\ln t}/2}-1)} \tag{25}$$

$$\lambda_g = \frac{\mu_g}{g_n} \quad \& \quad V_g = \frac{\sigma_g}{\mu_g} \tag{26}$$

Fig. 3 shows the difference between the evaluated mean and coefficient of variation of the function g calculated by the proposed method and by the MCS method plotted against steel strain at the extreme tensile rebar. A concrete compressive strength of 40MPa is used for these results. The results show that the difference is not significant, especially for high strain at extreme tensile rebar, where the applied approximation for variable g becomes more accurate. The coefficient of variation shows higher disagreement between the MCS and the proposed method. However, because of the low coefficient of variation, this disagreement is of less importance.

The bias factor and coefficient of variation of the function *g* can now be used in the estimation of the probability density function of the curvature ductility. It is assumed that all variables are uncorrelated. There is a small correlation between *g* and the rebar percentage. However, this correlation is negligible. Again, using the central limit theorem, the curvature ductility as a random variable follows lognormal distribution. Thus, its nominal, bias factor and coefficient of variation are evaluated as shown in Eqs. (27) to (29).

$$\mu_{\varphi n} = \frac{\alpha_{1n}\beta_{1n}f'_{cn}E_{sn}\varepsilon_{cun}b_n d_n g_n}{f_{yn}^2 A_{sn}} \tag{27}$$

$$\lambda_{\mu_\varphi} = \frac{\lambda_{\alpha_1}\lambda_{\beta_1}\lambda_{f'_c}\lambda_{E_s}\lambda_{\varepsilon_{cu}}\lambda_b\lambda_d\lambda_g}{\lambda_{f_y}^2\lambda_{A_s}} \tag{28}$$

$$V_{\mu_\varphi} = \sqrt{V_{\alpha_1}^2 + V_{\beta_1}^2 + V_{f'_c}^2 + V_{E_s}^2 + V_{\varepsilon_{cu}}^2 + V_b^2 + V_d^2 + V_g^2 + 4V_{f_y}^2 + V_{A_s}^2} \tag{29}$$

In Eq. (28), it can be seen that the yield strength of steel appears with a multiplier of 4, which shows its relative importance among the random variables in evaluating the probability model of curvature ductility. The curvature ductility follows the lognormal distribution, regardless of the probability density function of the contributing random variables. Therefore, of main random variables, only the nominal, bias factor and the coefficient of variation are required for evaluating the probability density function of curvature ductility.

In order to compare the above with the results of the MCS method, Fig. 4 depicts the variations in the means and the coefficients of variation of curvature ductility with respect to the extreme tensile rebar. As expected, perfect agreement exists between the assumed lognormal distribution and the MCS results. The results show that the probability density functions of all contributing random variables are not important, and that knowing only the nominal values, bias factors and coefficient of variation is adequate. The approximation in the form of *g* does not show any appreciable effect on the results and, as mentioned previously, because of the low coefficient of variation of the *g* function, this approximation is reasonable. According to this result, the curvature ductility is well modelled by the lognormal distribution and its statistical properties are calculated as presented in the above.

4.2 Moment redistribution

At this stage, using Eq. (11), the probability density function of the moment redistribution factor can be evaluated based on the probability density function of the curvature ductility and the plastic hinge length. It was found that the curvature ductility follows lognormal distribution. Here, we have a linear function of moment curvature, $m = \mu_\varphi - 1$. The probability density function of *m* is not lognormal. However, the mean and the standard deviation of function *m* can be calculated as shown in Eq. (30).

$$\mu_m = \mu_{\mu_\varphi} - 1 \quad \& \quad \sigma_m = \sigma_{\mu_\varphi} \tag{30}$$

The exact probability density function of *m* may be evaluated using the monotonic and one-to-one behaviour of linear transformation. Although the function *m* does not follow lognormal distribution, it could be approximated well using this distribution. Eqs. (31) and (32) show the exact and approximated probability density functions of *m*, respectively.

$$f_M(m) = \frac{1}{(m+1)\sigma_{\ln \mu_\varphi}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln(m+1) - \mu_{\ln \mu_\varphi}}{\sigma_{\ln \mu_\varphi}}\right)\right] \tag{31}$$

$$f_M(m) = \frac{1}{m\sigma_{\ln m}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln m - \mu_{\ln m}}{\sigma_{\ln m}}\right)\right] \tag{32}$$

Fig. 5 illustrates both the exact and the approximated probability density functions of *m* for a wide range of strains at the extreme tensile rebar. The curve with the highest density is for low strain (here, 0.0075), while the curve with the lowest density belongs to high strain at the extreme tensile rebar (here, 0.025).

As is evident from Fig. 5, when the strain at the extreme tensile rebar increases (which is synonymous with increasing the curvature ductility), function *m* could be accurately approximated by the lognormal distribution. The error in this approximation is negligible, especially when the strain at the extreme tensile rebar becomes larger. It is worth mentioning that, because of high ductility, the high-strain region of the extreme tensile rebar is of more influence in the moment redistribution of RC beams.

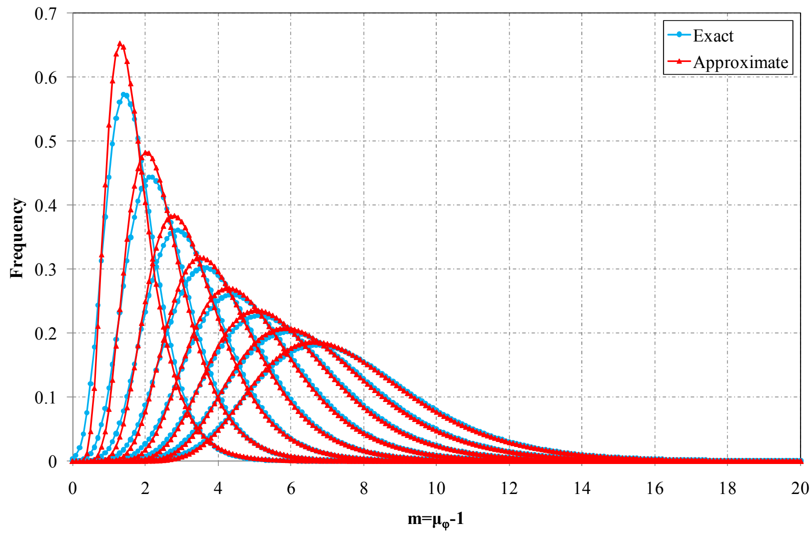


Fig. 5 Comparison of exact and approximated probability density function of function *m*

Statistical data on the plastic hinge length are rare. In this study, the model proposed by Lu and Gu (2004) is used, in which they proposed a normal distribution for the plastic hinge length. In their study, the coefficient of variation of the model error was found to be around 0.2. For small coefficients of variation, the normal distribution can be approximated using lognormal distribution. Here, using this approximation, and knowing that $\mu_\varphi - 1$ approximately follows lognormal distribution, the product of the normalised plastic hinge length (with respect to span length) and $\mu_\varphi - 1$ also follows lognormal distribution. By defining a new variable, Eq. (11) is rewritten as Eq. (33).

$$\beta = 1 - \frac{1}{1 + 2\left(\frac{l_p}{l}\right)\left(\mu_\varphi - 1\right)} = 1 - \frac{1}{1+x} = \frac{x}{x+1} \tag{33}$$

The variable x and its mean and standard deviation are calculated as shown in Eqs. (34) to (36).

$$x = 2\left(\frac{l_p}{l}\right)\left(\mu_\varphi - 1\right) = 2pm \tag{34}$$

$$\mu_{\ln x} = \ln(2) + \mu_{\ln p} + \mu_{\ln m} = \ln\left[\frac{2\mu_p \mu_m}{\sqrt{(1+V_p^2)(1+V_m^2)}}\right] \tag{35}$$

$$\sigma_{\ln x} = \sqrt{\sigma_{\ln p}^2 + \sigma_{\ln m}^2} = \sqrt{\ln\left[(1+V_p^2)(1+V_m^2)\right]} \tag{36}$$

Eq. (32) shows that β is a one-to-one and monotonically increasing function with respect to variable x used above. Using this property of function β, its probability density function and cumulative density function can easily be calculated as Eqs. (37) and (38) (Benjamin and Cornell, 1975).

$$f_\beta(\beta) = \frac{1}{(1-\beta)^2} f_X\left(\frac{\beta}{1-\beta}\right) = \frac{1}{(1-\beta)^2 \sigma_{\ln x}} \phi\left[\frac{\ln\left(\frac{\beta}{1-\beta}\right) - \mu_{\ln x}}{\sigma_{\ln x}}\right] \quad 0 < \beta < 1 \tag{37}$$

$$F_\beta(\beta) = F_X\left(\frac{\beta}{1-\beta}\right) = \Phi\left[\frac{\ln\left(\frac{\beta}{1-\beta}\right) - \mu_{\ln x}}{\sigma_{\ln x}}\right] \quad 0 < \beta < 1 \tag{38}$$

In Eqs. (36) and (37), functions ϕ and Φ represent the probability density function and cumulative density function of standard normal distribution, respectively. These equations show that the probability density functions of the moment redistribution factor can be evaluated using simple standard normal distribution. The entire procedure shown above for finding a closed-form expression for the moment redistribution factors of RC beams is derived from the central limit theorem, based on the monotonically increasing behaviour of the derived function, which relates the moment redistribution, the plastic hinge length and the curvature ductility.

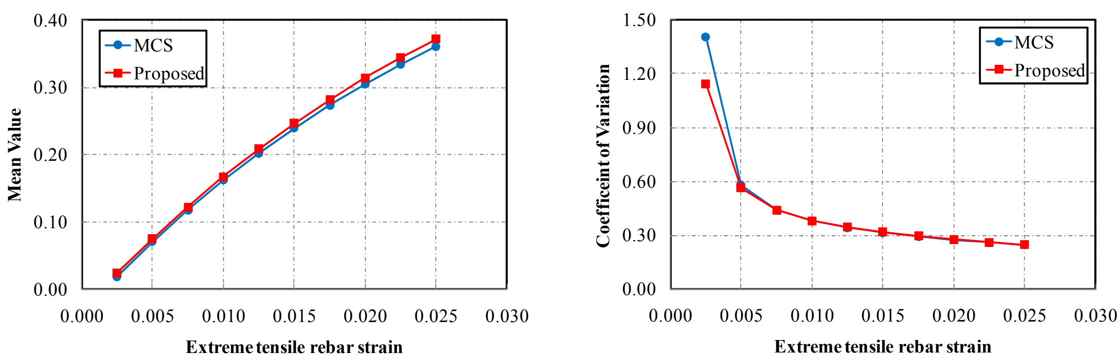
The point of maximum likelihood of function β is not too distant from its mean, and the behaviour of this function could be linearly approximated around the mean point. First Order Second Moment (FOSM) approximation might be used in order to find its mean and standard deviation, as is seen in Eqs. (39) to (41).

$$E(\beta) \approx \frac{\mu_x}{1+\mu_x} = \frac{e^{\mu_{\ln x}+\sigma_{\ln x}}}{1+e^{\mu_{\ln x}+\sigma_{\ln x}}} \tag{39}$$

$$Var(\beta) \approx \left[\frac{d\beta}{dx}\right]^2_{\mu_x} Var(x) = \frac{1}{(1+\mu_x)^2}\sigma_x^2 = \frac{(e^{2\mu_{\ln x}+\sigma^2_{\ln x}}\sqrt{e^{\sigma^2_{\ln x}}-1})}{(1+e^{\mu_{\ln x}+\frac{1}{2}\sigma^2_{\ln x}})^2} \tag{40}$$

$$V_\beta = \frac{\sigma_\beta}{\mu_\beta} = \frac{V_x}{1+\mu_x} = \frac{\sqrt{e^{\sigma^2_{\ln x}}-1}}{1+e^{\mu_{\ln x}+\frac{1}{2}\sigma^2_{\ln x}}} \tag{41}$$

A comparison of the mean and coefficient of variation values of the β function derived by the MCS and the proposed method is shown in Fig. (6). The results in this figure show perfect agreement between the proposed method and the MCS. The linear approximation of the β function about the mean value shows a very good agreement with the result obtained from the MCS, with a high simulation number. The span-to-depth ratio of 20 and a normalised plastic hinge of 0.0358 are used in Fig. 6 to find the expectations of the β function.



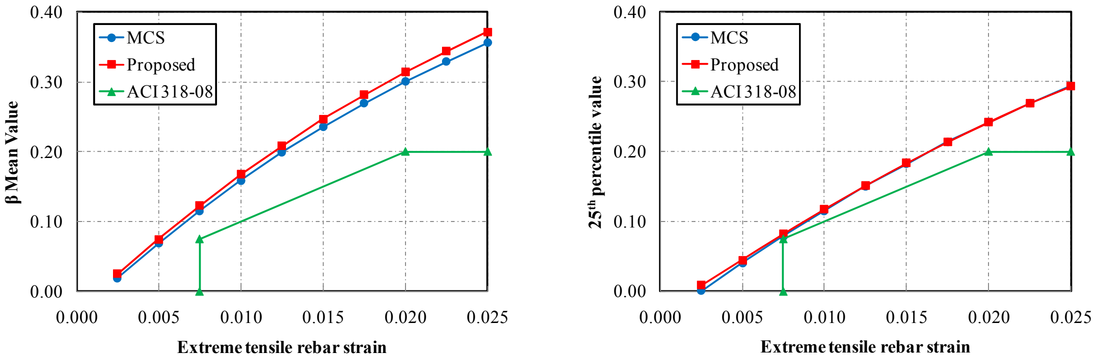
(a) Mean value (b) Coefficient of variation

Fig. 6 Comparison of proposed and MCS methods for mean and coefficient of variation of moment redistribution factor

5. Worked example

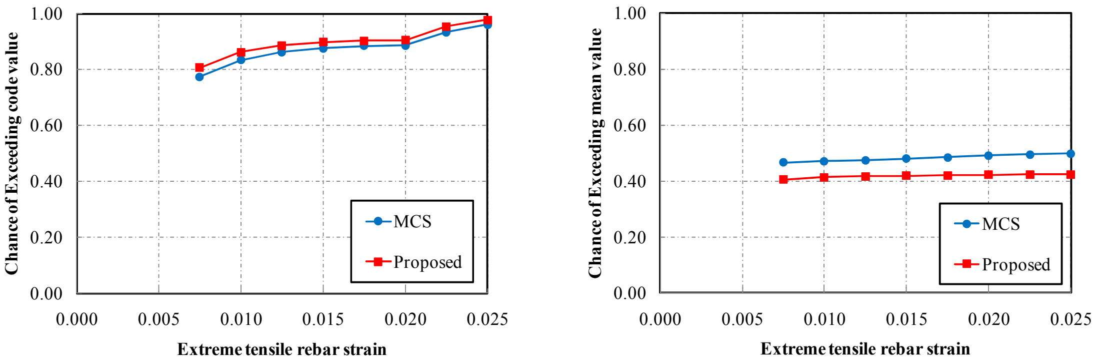
To show the efficiency of the proposed method, a design case is considered in this section. A fixed RC beam with the span-to-depth ratio of 20 and a normalised plastic hinge of 0.035 is considered. This case is near the lower bound assumptions considered in the design codes for evaluating moment redistribution factors. The nominal compressive strength of concrete is assumed to be 40MPa, and G60 rebar steel with a nominal yield strength of 420MPa is used. All of the nominal values of the material-related random variables, like the modulus of elasticity of steel and concrete and concrete ultimate strain, are taken from ACI 318 (2011) design code. Other statistical data for all random variables in this study are shown in Table 1.

The described closed-form procedure is used, along with the MCS method. Fig. 7 shows the mean value and the 25th percentile value (which has 75% chance of being exceeded) for the moment redistribution factor, as well as the code-specified moment redistribution factors. Good agreement exists between the MCS and the proposed method for both mean and 25th percentile values.



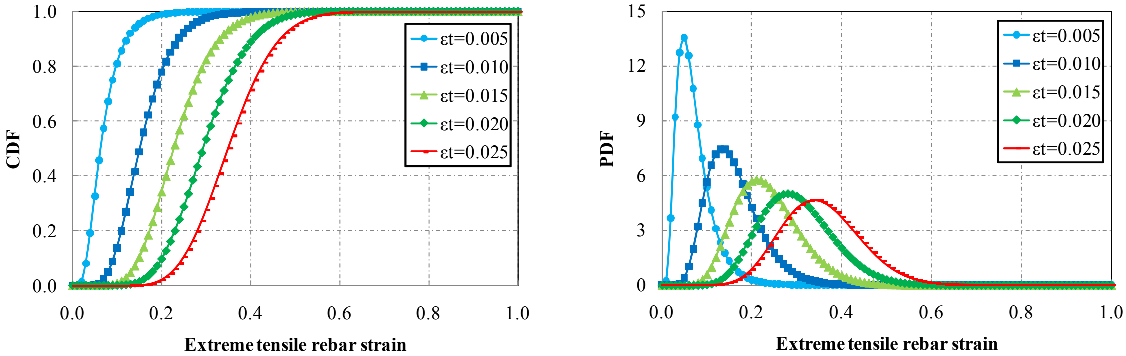
(a) Mean value (b) 25th percentile value

Fig. 7 Comparison of MCS and proposed method for mean and 25th percentile values of moment redistribution factor



(a) Based on code-specified value (b) Based on mean value

Fig. 8 Comparison of MCS and proposed method for chance of being exceeded



(a) CDF (b) PDF

Fig. 9 Probabilistic distributions of moment redistribution factor using proposed method

It may be useful to evaluate the reliability of the code-specified moment redistribution factors. In Fig. 7, the 25th percentile values of moment redistribution factors were shown for different extreme tensile rebar strains. The reliability of the ACI 318 specified moment redistribution factors are now plotted against extreme tensile rebar strain. Based on ACI 318 code, the moment redistribution factor shown in Fig. 7 is calculated as shown in Eq. (42).

$$\beta = 10\varepsilon_t \leq 0.2 \tag{42}$$

In Eq. (42), ε_t is the strain at the extreme tensile rebar. This strain must be greater than 0.0075. In Fig. 8, the chances of the mean and code-specified values being exceeded are illustrated for both the MCS and the proposed method. It is worth mentioning that, in the MCS method, the normalised plastic hinge length follows normal distribution, while the proposed closed-form method assumes that this variable follows lognormal distribution. The results for code-specified values of the moment redistribution factor are quite close, while the agreement is not so close for the mean value. In Fig. 8(b), the mean value used for probability estimation is not the same. As can be seen in Fig. 7(a), there is a small difference (in the order of 5%) between the MCS and the proposed method in estimating the mean value. Therefore, the error accumulates in finding the probability of exceedance for the mean value, because of the error in the mean value itself and in the probability density function.

The probability density function (PDF) and cumulative density function (CDF) of the β function can be easily plotted using Eqs. (37) and (38). Fig. 9 illustrates the PDFs and CDFs of the β function for a range of extreme tensile strains of rebar. The code-specified values are shown on the CDF curve. Even though the current use of high-speed computers allows large numbers of simulations to be completed in a few seconds, closed-form solutions for the PDF and CDF of any function can provide a deeper understanding of the role of each variable.

This case study has shown that the proposed closed-form solution probabilistic evaluation of the moment redistribution factor provides reliable results. In addition to being quick, this method provides a deeper understanding of the important random variables in the probability models of curvature ductility and moment redistribution factor.

6. Conclusions

The curvature ductility and permissible moment redistribution, like other design parameters in reinforced concrete mechanics, should be calculated using a probabilistic approach. In a reliability-based analysis, the characteristic values (with their specific chance of being exceeded) of permissible moment redistribution factors would be evaluated (instead of the lower bound values) and this improves reliability.

In this study, a simple expression was found for the evaluation of curvature ductility in RC beams of singly-reinforced rectangular section. The concept of equating demand and capacity rotations was then used in the evaluation of the allowable moment redistribution in continuous RC beams. For simplicity, a beam fixed at both ends was considered. It was assumed that this beam could represent the interior span of a continuous beam. A probabilistic approach was then used to find the mean, variance and probability model for both curvature ductility and moment redistribution factor.

The probabilistic approach led to a simple closed-form solution for curvature ductility and moment redistribution factor. This simple derivation of the probabilistic model for curvature ductility and moment redistribution factors arises from central limit theorem and from the monotonic behaviour of moment redistribution function with respect to curvature ductility and plastic hinge length. The output of the proposed method was checked against the MCS method. Although some minor simplifications were used in the direct probabilistic method, there was very good agreement between the MCS and proposed method. The proposed method can be used in any future probabilistic evaluation of curvature ductility and moment redistribution factors.

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