

Bending of a cracked functionally graded nanobeam

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(Received March 29, 2017, Revised May 24, 2018, Accepted July 17, 2018)

Abstract. In this study, static bending of an edge cracked cantilever nanobeam composed of functionally graded material (FGM) subjected to transversal point load at the free end of the beam is investigated based on modified couple stress theory. Material properties of the beam change in the height direction according to exponential distributions. The cracked nanobeam is modelled using a proper modification of the classical cracked-beam theory consisting of two sub-nanobeams connected through a massless elastic rotational spring. The inclusion of an additional material parameter enables the new beam model to capture the size effect. The new non-classical beam model reduces to the classical beam model when the length scale parameter is set to zero. The considered problem is investigated within the Euler-Bernoulli beam theory by using finite element method. In order to establish the accuracy of the present formulation and results, the deflections are obtained, and compared with the published results available in the literature. Good agreement is observed. In the numerical study, the static deflections of the edge cracked FGM nanobeams are calculated and discussed for different crack positions, different lengths of the beam, different length scale parameter, different crack depths, and different material distributions. Also, the difference between the classical beam theory and modified couple stress theory is investigated for static bending of edge cracked FGM nanobeams. It is believed that the tabulated results will be a reference with which other researchers can compare their results.

Keywords: open edge crack; modified couple stress theory; functionally graded materials; nanobea

1. Introduction

With the great advances in technology in recent years, micro and nano structures have found many applications. In these structures, micro beams and micro tubes are widely used in micro- and nano electromechanical systems (MEMS and NEMS) such as sensors (Zook *et al.* 1992, Pei *et al.* 2004), actuators (Senturia 1998, Rezazadeh *et al.* 2006). In investigation of micro and nano structures, the classical continuum mechanics is not capable of explanation of the size-dependent behaviors. Nonclassical continuum theories such as higher order gradient theories and the couple stress theory are capable of explanation of the size dependent behaviors which occur in micro/nano-scale structures.

Functionally graded materials (FGMs) are a new generation of composites where the volume fractions of the FGM constituents vary gradually, giving a non-uniform nanostructure with continuously graded macro properties such as elasticity modulus, density, heat conductivity, etc.. With the advance of the material science, micro–electro-mechanical systems are the new field in

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which FGMs have been utilized to achieve the desired performance. With the development of technology, FGMs are initialized to be used in micro/nano structural systems.

In recent years, functionally graded materials have been widely used in micro/nanoelectromechanical

gradient systems (MEMS/NEMS), such as the components in the form of shape memory alloy thin films with a global thickness in micro- or nano-scale (Fu *et al.* 2003, Witvrouw and Mehta 2005, Lü *et al.* 2009), electrically actuated MEMS devices (Hasanyan *et al.* 2008, Mohammadi-Alasti *et al.* 2011, Zhang and Fu 2012), and atomic force microscopes (AFMs) (Rahaeifard *et al.* 2009). In all these applications, the thickness of the microstructures (i.e., microbeams or microplates) is typically on the order of microns and sub-microns. At the present time, the experimental investigations of the micro materials are still a challenge because of difficulties confronted in the micro scale. Therefore, mechanical theories and atomistic simulations have been used for micro structural analysis. The process of the atomistic simulations is very difficult and takes much time. So, continuum theory is the most preferred method for the analysis of the micro and nano structures. Classical continuum mechanics does not contain the size effect, because of its scale-free character. The nonlocal continuum theory initiated by Eringen (1972) has been widely used to mechanical behavior of nano-micro structures.

The size effect plays an important role on the mechanical behavior of micro/nano structures at the nanometer scale that the classic theory has failed to consider when the size reduces from macro to nano (Toupin 1962, Mindlin and Tiersten 1962, Mindlin 1963, Fleck and Hutchinson 1993, Yang *et al.* 2002, Lam *et al.* 2003). Therefore, higher-order theories modified couple stress theory and modified strain re used in the mechanical model of the nano-micro structures (Yang *et al.* 2002, Lam *et al.* 2003).

The determination of the nano-structural material length scale parameters is very difficult experimentally. So, Yang *et al.* (2002) proposed the modified couple stress theory in which the strain energy has been shown to be a quadratic function of the strain tensor and the symmetric part of the curvature tensor, and only one length scale parameter is included. After this, the modified couple stress and the strain gradient elasticity theories have been widely applied to static and dynamic analysis of beams (Park and Gao 2006, Ma *et al.* 2008, Kong *et al.* 2008, Asghari *et al.* 2010, Wang 2010, Şimşek 2010, Kahrobaiyan *et al.* 2010, Xia *et al.* 2010, Ke *et al.* 2011, Akgöz and Civalek 2012, Ansari *et al.* 2012, Gürses *et al.* 2012, Wang *et al.* 2013, Kocatürk and Akbaş 2013, Kong 2013, Ghayesh *et al.* 2013, Daneshmehr *et al.* 2013, Akgöz and Civalek 2013). More recently, Darijani and Mohammadabadi (2014) proposed a new deformation beam theory for static and dynamic analysis of microbeams which includes unknown functions takes into account shear deformation and satisfies both of shear and couple-free conditions on the upper and lower surfaces of the beam based on a modified couple stress theory. Tang *et al.* (2014) analyzed a theoretical model for flexural vibrations of microbeams in flow with clamped-clamped ends based on a modified couple stress theory. Sedighi *et al.* (2014) investigated the dynamic pull-in instability of vibrating micro-beams undergoing large deflection under electrostatically actuation. Farokhi and Ghayesh (2015) studied the three-dimensional motion characteristics of perfect and imperfect Timoshenko microbeams under mechanical and thermal forces based on the modified couple stress theory. Ansari *et al.* (2015) studied an exact solution of vibrations of postbuckled microscale beams based on the modified couple stress theory. Dai *et al.* (2015) developed a new nonlinear theoretical model for cantilevered microbeams and to explore the nonlinear dynamics based on the modified couple stress theory, taking into account of one single material length scale parameter. Farokhi and Ghayesh (2015) investigated the three-dimensional

motion characteristics of perfect and imperfect Timoshenko microbeams under mechanical and thermal forces.

In recent years, various investigations have been carried out to study of static and vibration behavior of the Functionally graded (FGM) micro/nano beams; (Ansari *et al.* 2011, 2013a, b, Liu and Reddy 2011, Reddy 2011, Ke *et al.* 2011, Salamat-Talab *et al.* 2012, Nateghi and Salamat-Talab 2013, Akgöz and Civalek 2013, Şimşek and Reddy 2013a, Zamanzadeh *et al.* 2013, Şimşek *et al.* 2013, Ansari *et al.* 2014a, b), buckling (Ansari *et al.* 2012, 2013c, d, Şimşek and Reddy 2013b, Wang *et al.* 2014, Akgöz and Civalek 2014, 2015a, b, 2016) of the FGM beams in which the nonclassical continuum mechanics have been employed. Berrabah *et al.* (2013) examined bending, buckling and free vibration of nanobeams by using unified nonlocal shear deformation theory. Tagrara *et al.* (2015) presented a trigonometric refined beam theory for the bending, buckling and free vibration analysis of carbon nanotube-reinforced composite beams resting on elastic foundation. Zemri *et al.* (2015) investigated nonlocal shear deformation beam theory for bending, buckling, and vibration of functionally graded nanobeams using the nonlocal differential constitutive relations of Eringen. Aissani *et al.* (2015) presented a new nonlocal hyperbolic shear deformation beam theory for the static, buckling and vibration of nanoscale-beams embedded in an elastic medium. Akbaş (2016b) investigated forced vibration of viscoelastic nano scale beams. Akbaş (2016c) presented static analysis of nanoplates by using differential quadrature method. Ehyaei *et al.* (2016) examined effect on free vibration characteristics of functionally graded size-dependent nanobeams by using a semi analytical differential transform method. Ebrahimi and Barati (2016a, b) investigated buckling of functionally graded nanobeams. Eltaher *et al.* (2016) studied the effects of thermal load and shear force on the buckling of nanobeams. Akbaş (2017a) analyzed forced vibration of functionally graded nanobeams.

Micro and nano structures are subjected to destructive effects in the form of initial defects within the material or caused by fatigue or stress concentration, for example, crack occurrence in ZnO nano-rods is due to thermal fabrication process (Fang and Chang 2003, Fang *et al.* 2003). As a result of destructive effects, cracks occur in the structural elements. It is known that cracks cause local flexibility and changes in structural stiffness. Understanding the mechanical behaviour edge-cracked structures and detection of cracks are very important for safety of nano structures.

In the literature, the studies of the cracked micro-nano structures are as follows; Loya *et al.* (2009) studied the flexural vibrations of cracked micro- and nanobeams with the rotational and linear spring model based on nonlocal elasticity. Hasheminejad *et al.* (2011) investigated the flexural vibrations of cracked micro- and nanobeams in the presence of surface effects with the rotational linear spring model. Hsu *et al.* (2011) analysed longitudinal vibration of cracked nanobeams based on nonlocal elasticity theory. Torabi and Nafar Dastgerdi (2012) studied with the free transverse vibration of cracked nanobeams modeled based on Eringen's nonlocal elasticity theory and Timoshenko beam theory with a rotational spring. Liu *et al.* (2013) investigated Vibration behavior of a cracked Euler-Bernoulli micro-cantilever beam under coupling action of nonlinear electrostatic force and squeeze film damping effect. Hosseini-Hashemi *et al.* (2014) investigated the dynamics of thin and thick cracked nanobeams with surface effects. Tadi Beni *et al.* (2015) examined the transverse vibration of cracked nano-beam based on modified couple stress theory with rotational spring model. Akbaş (2016a) investigated static analysis of cracked micro beams. Shaat *et al.* (2016) investigated vibration analysis of Cracked Nano- Beams made of Nanocrystalline Materials. Akbaş (2017b) studied free vibration of edge cracked functionally graded microbeams. Akbaş (2018) investigated forced vibration analysis of cracked functionally graded microbeams.

This paper examines static bending of an edge cracked FGM cantilever nanobeam based on the modified couple stress theory. The nanobeam is subjected to transversal point load at the free end of the beam. Material properties of the beam change in the height direction according to exponential distributions. The cracked beam is modelled using a proper modification of the classical cracked-beam theory consisting of two sub-beams connected through a massless elastic rotational spring. The considered problem is investigated within the Euler-Bernoulli beam theory by using finite element method. This nanobeam model incorporates the material length scale parameter which can capture the size effect. In order to establish the accuracy of the present formulation and results, the deflections are obtained, and compared with the published results available in the literature. Good agreement is observed. In the numerical study, the static deflections of the edge cracked FGM nanobeams are calculated and discussed for different crack positions, different lengths of the beam, different length scale parameter, different crack depths, and different material distributions. Also, the difference between the classical beam theory and modified couple stress theory is investigated for static bending of edge cracked FGM nanobeams. It is believed that the tabulated results will be a reference with which other researchers can compare their results.

2. Theory and formulations

Consider a cantilever FGM nanobeam of length L , width b , height h , containing an edge crack of depth a located at a distance L_1 from the left end, as shown in Fig. 1. It is assumed that the crack is perpendicular to beam surface and always remains open. The beam is subjected to a point load (Q) in the transverse direction as seen from Fig. 1.

The material properties of the FGM nanobeam P , i.e., Young's modulus E , Poisson's ratio ν and shear modulus G vary continuously in the thickness direction (Y axis) according to exponential distributions as follows

$$P(Y) = P_0 e^{\beta Y} \quad (1)$$

where P_0 is the material properties at the midplane ($Y = 0$) of the beam. β is a constant characterizing the gradual variation of the material properties along thickness direction. When $\beta = 0$, the material of the beam is homogeneous. According to Eq. (1) that when $Y = h/2$, $P = P_B$ (P_B is the material properties of the bottom). When $Y = -h/2$, $P = P_T$ (P_T is the material properties of the top).

2.1 The modified couple stress theory

The strain energy density for a linear elastic material which is a function of both strain tensor

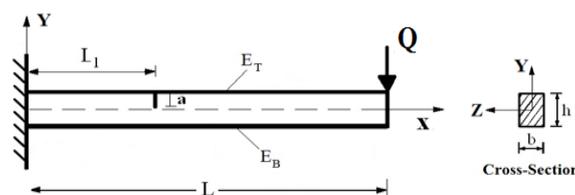


Fig. 1 A cantilever FGM nanobeam with an open edge crack subjected to a point load

and curvature tensor is introduced by Yang *et al.* (2002) for the modified couple stress theory

$$U = \int_V (\sigma : \varepsilon + m : \chi) dV \quad (2)$$

where σ is the stress tensor, ε is the strain tensor, m is the deviatoric part of the couple stress tensor, χ is the symmetric curvature tensor, defined by

$$\sigma = \lambda \operatorname{tr}(\varepsilon)I + 2\mu\varepsilon \quad (3)$$

$$\varepsilon = \frac{1}{2}[\nabla u + (\nabla u)^T] \quad (4)$$

$$m = 2l^2\mu\chi \quad (5)$$

$$\chi = \frac{1}{2}[\nabla\theta + (\nabla\theta)^T] \quad (6)$$

where λ and μ are Lamé's constants, l is a material length scale parameter which is regarded as a material property characterizing the effect of couple stress. The material length scale is a size effect parameter which has important role on the structural behavior of micro/nano structures, because the dimensions of these structures are smaller than the atomic/molecular distances. However, classical continuum mechanics does not contain the material length scale parameter, because of its scale-free character. u is the displacement vector and θ is the rotation vector, given by

$$\theta = \frac{1}{2} \operatorname{curl} u \quad (7)$$

The parameters λ and μ in the constitutive equation are given by

$$\lambda(Y) = \frac{E(Y)\nu(Y)}{(1+\nu(Y))(1-2\nu(Y))}, \quad \mu(Y) = \frac{E(Y)}{2(1+\nu(Y))} \quad (8)$$

Where E is the modulus of elasticity and ν is the Poisson's ratio and their dependence on Y coordinate are given by Eq. (1).

2.2 Governing equations of FGM nanobeams

According to the coordinate system (X, Y, Z) shown in Fig. 1, based on Euler-Bernoulli beam theory, the axial and the transverse displacement field are expressed as

$$u(X, Y) = u_0(X) - Y \frac{\partial v_0(X)}{\partial X} \quad (9)$$

$$v(X, Y, t) = v_0(X, t) \quad (10)$$

$$w(X, Y, t) = 0 \quad (11)$$

where u , v , w are x , y and z components of the displacements, respectively. Also, u_0 and v_0 are the axial and the transverse displacements in the mid-plane.

Because the transversal surfaces of the beam is free of stress, then

$$\sigma_{zz} = \sigma_{yy} = 0 \quad (12)$$

By using Eqs. (3), (8) and (9) and strain- displacement relation can be obtained

$$\varepsilon_{xx} = \frac{\partial u}{\partial X} = \frac{\partial u_0(X)}{\partial X} - Y \frac{\partial^2 v_0(X)}{\partial X^2} \quad (13a)$$

$$\varepsilon_{yy} = \varepsilon_{zz} = Y \frac{v(Y) \partial^2 v_0(X)}{\partial X^2} \quad (13b)$$

$$\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{xy} = 0 \quad (13c)$$

By using Eqs. (7), (9), (10) and (11)

$$\theta_z = \frac{\partial v_0(X)}{\partial X}, \quad \theta_x = \theta_y = 0 \quad (14)$$

Substituting Eq. (14) into Eq. (6), the curvature tensor χ can be obtained as follows

$$\chi_{xz} = \frac{1}{2} \frac{\partial^2 v_0(X)}{\partial X^2}, \quad \chi_{xx} = \chi_{xy} = \chi_{yy} = \chi_{yz} = \chi_{zz} = 0 \quad (15)$$

According to Hooke's law, constitutive equations of the FGM nanobeam are as follows

$$\sigma_{xx} = E(Y) \varepsilon_{xx} = E(Y) \left[\frac{\partial u_0(X)}{\partial X} - Y \frac{\partial^2 v_0(X)}{\partial X^2} \right] \quad (16)$$

Where σ_{xx} and ε_{xx} are normal stresses and normal strains in the X direction, respectively. Substituting Eq. (15) into Eq. (5), the couple stress tensor can be obtained as follows

$$m_{xz} = l^2 \mu(Y) \frac{1}{2} \frac{\partial^2 v_0(X)}{\partial X^2} \quad (17a)$$

$$m_{xx} = m_{xy} = m_{yy} = m_{yz} = m_{zz} = 0 \quad (17b)$$

where μ is shear modulus which is defined by Eq. (8). Based on Euler-Bernoulli beam theory, the elastic strain energy (U_i) of the FGM nanobeam is expressed as

$$U_i = \frac{1}{2} \int_0^L \int_A (\sigma_{ij} \varepsilon_{ij} + m_{ij} \chi_{ij}) dA dX \quad (18)$$

By substituting Eqs. (16), (17), (13a) and (15) into Eq. (18), elastic strain energy (U_i) can be rewritten as follows

$$U_i = \frac{1}{2} \int_0^L \left[A_{11} \left(\frac{\partial u_0}{\partial X} \right)^2 - 2B_{11} \left(\frac{\partial u_0}{\partial X} \right) \left(\frac{\partial^2 v_0}{\partial X^2} \right) + D_{11} \left(\frac{\partial^2 v_0}{\partial X^2} \right)^2 + \frac{1}{4} l^2 A_{55} \left(\frac{\partial^2 v_0}{\partial X^2} \right)^2 \right] dX \quad (19)$$

Where

$$(A_{11}, B_{11}, D_{11}) = \int_A E(Y)(1, Y, Y^2) dA, \quad A_{55} = \int_A \mu(Y) dA \quad (20)$$

The potential energy of the external load can be written as

$$W = Q v_0(X = L) \quad (21)$$

The considered problem is solved by using finite element method.

Total nodal displacements q which is written for a two-node beam element, each node has three degrees of freedom, shown in Fig. 2 are defined as follows

$$\{q\}_e = \left[u_i^{(e)}, v_i^{(e)}, \theta_i^{(e)}, u_j^{(e)}, v_j^{(e)}, \theta_j^{(e)} \right]^T \quad (22)$$

The displacement field of the finite element is expressed in terms of nodal displacements as follows

$$u^{(e)}(X) = \varphi_1^{(U)}(X) u_i + \varphi_2^{(U)}(X) u_j = [\varphi^{(U)}] \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = [\varphi^{(U)}] \{q\}_U \quad (23)$$

$$v^{(e)}(X) = \varphi_1^{(V)}(X) v_i + \varphi_2^{(V)}(X) \theta_i + \varphi_3^{(V)}(X) v_j + \varphi_4^{(V)}(X) \theta_j$$

$$[\varphi^{(V)}] \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} = [\varphi^{(V)}] \{q\}_V \quad (24)$$

Where u_i , v_i and θ_i are axial displacements, transverse displacements and slopes at the two end nodes of the beam element, respectively. $j_i^{(U)}$ and $j_i^{(V)}$ are Hermitian shape functions for axial and transverse degrees of freedom, respectively, which are given in Appendix.

By substituting Eqs. (23) and (24) into Eq. (19), energy functions can be rewritten as follows

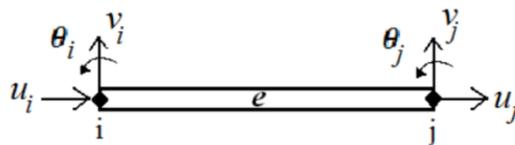


Fig. 2 A two-node beam element

$$U_i = \frac{1}{2} \int_0^L \left[A_{11} \left(\frac{\partial \varphi^{(U)}}{\partial X} \{q\}_U \right)^2 - 2B_{11} \left(\frac{\partial \varphi^{(U)}}{\partial X} \{q\}_U \right) \left(\frac{\partial^2 \varphi^{(V)}}{\partial X^2} \{q\}_V \right) + D_{11} \left(\frac{\partial^2 \varphi^{(V)}}{\partial X^2} \{q\}_V \right)^2 + \frac{1}{4} l^2 A_{55} \left(\frac{\partial^2 \varphi^{(V)}}{\partial X^2} \{q\}_V \right)^2 \right] dX \quad (25)$$

The total potential energy of the beam is given by

$$\Pi = U - W \quad (26)$$

According to the minimum total potential energy principle, nodal displacements $\{q\}$ which correspond to the minimum of the total potential energy are determined by the conditions

$$\frac{\partial \Pi}{\partial q} = 0 \quad (27)$$

Differentiation of Π in respect to nodal displacements $\{q\}$ produces the following equilibrium equations for a finite element

$$[K]\{q\} = \{F\} \quad (28)$$

where, $[K]$ is the stiffness matrix and $\{F\}$ is the load vector. The components of the stiffness matrix $[K]$

$$[K] = \begin{bmatrix} [K^A] & [K^B] \\ [K^B]^T & [K^D] \end{bmatrix} \quad (29)$$

Where

$$[K^A] = \int_0^{L_e} A_{11} \left[\frac{\partial \varphi^{(U)}}{\partial X} \right]^T \left[\frac{\partial \varphi^{(U)}}{\partial X} \right] d \quad (30a)$$

$$[K^B] = - \int_0^{L_e} B_{11} \left[\frac{\partial^2 \varphi^{(V)}}{\partial X^2} \right]^T \left[\frac{\partial \varphi^{(U)}}{\partial X} \right] d \quad (30b)$$

$$[K^D] = \int_0^{L_e} \left(D_{11} + \frac{1}{4} l^2 A_{55} \right) \left[\frac{\partial^2 \varphi^{(V)}}{\partial X^2} \right]^T \left[\frac{\partial^2 \varphi^{(V)}}{\partial X^2} \right] d \quad (30c)$$

Where L_e indicates the length of the finite beam element.

The load vector $\{F\}$ is expressed as

$$\{F\} = \{\varphi(X)\}^T Q \quad (31)$$

2.3 Crack modeling

The cracked FGM nanobeam is modelled using a proper modification of the classical cracked-beam theory consisting of two sub-beams connected through a massless elastic rotational spring

shown in Fig. 3.

The bending stiffness of the cracked section k_T is related to the flexibility G by

$$k_T = \frac{1}{G} = 0 \tag{32}$$

The cracked section's flexibility G can be derived from Broek's approximation (Broek 1986)

$$\frac{(1 - \nu^2)K_I^2}{E(a)} = \frac{M^2}{2} \frac{dG}{da} \tag{33}$$

where M is the bending moment at the cracked section, a is the crack depth, K_I is the stress intensity factor (SIF) under mode I bending load (which is a function of the geometry, loading, and material properties), and ν indicates Poisson's ratio (which is taken as constant since its influence on the SIF is quite limited (Erdogan and Wu 1997)). For an FGM strip with an open edge crack under bending, the analytical solution along with the expression of SIF has been given Yu and Chu (2009), which was obtained from the data given by Erdogan and Wu (1997) through the Lagrange interpolation technique

$$K_I = \frac{6M}{\pi b} \sqrt{\pi a} F(E_R, a/h) \tag{34}$$

where E_R is the ratio of Young's modulus at the bottom to the top surface of the beam (E_B/E_T), and F is an unknown function of two independent variables, given as (Yu and Chu 2009)

$$F(E_R, a/h) = \frac{p_1 + p_2 \ln(E_R) + p_3 [\ln(E_R)]^2 + p_4 [\ln(E_R)]^3 + p_5 (a/h) + p_6 (a/h)^2}{1 + p_7 \ln(E_R) + p_8 [\ln(E_R)]^2 + p_9 (a/h) + p_{10} (a/h)^2 + p_{11} (a/h)^3} \tag{35}$$

Here the coefficients $p_1, p_2, \dots, p_{10}, p_{11} = 1.1732, -0.3539, 0.0289, -0.0061, 0.6625, 3.072, -0.0014, -0.0017, 1.9917, -0.3496, -3.0982$ were obtained from Yu and Chu (2009), which were determined by fitting Eq. (33) using the least squares method to the numerical data of the SIF for specific material gradients and normalized crack sizes given by Erdogan and Wu (1997).

After substituting Eq. (34) into Eq. (33) and integrating, the flexibility coefficient of the crack section G is obtained as

$$G = \int_0^{a/h} \frac{72\pi (1 - \nu^2) a F^2 (E_R, a/h)}{E(a) h^4} da \tag{36}$$

The spring connects the adjacent left and right sub-elements, providing a jump between the

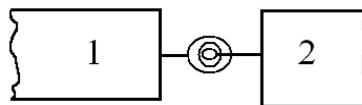


Fig. 3 Rotational spring model

slopes of the two FGM sub-elements at the crack location. In the massless spring model, the compatibility conditions enforce the continuities of the axial displacement, transverse deflection, axial force and bending moment across the crack at the cracked section ($X = L_1$), that is

$$u_1 = u_2, \quad v_1 = v_2, \quad N_1 = N_2, \quad M_1 = M_2 \quad (37)$$

A discontinuity exists in the slope at the cracked section

$$k_T \left(\frac{dv_1}{dX} - \frac{dv_2}{dX} \right) = k_T (\theta_1 - \theta_2) = M_1 \quad (38)$$

Based on the massless spring model, the stiffness matrix of the cracked section is

$$[K]_{(Cr)} = \begin{bmatrix} 1/G & -1/G \\ -1/G & 1/G \end{bmatrix} = \begin{bmatrix} k_T & -k_T \\ -k_T & k_T \end{bmatrix} \quad (39)$$

The stiffness matrix of the cracked section is written in terms of the displacement vector

$$\{q\}_{(Cr)} = \{\theta_1, \theta_2\}^T \quad (40)$$

where θ_1 and θ_2 are the angles on the two sides of the cracked section. With the addition of the crack model and by following the conventional assembly procedure for the finite elements, the system stiffness matrix is obtained as follows

$$[K]_{(S)} = [K] + [K]_{(Cr)} \quad (41)$$

where $[K]$ is the intact stiffness matrix for the intact FGM nanobeam, which appears in Eq. (29).

The dimensionless quantities can be expressed as

$$\bar{X} = \frac{X}{L}, \bar{Y} = \frac{Y}{L}, \bar{v} = \frac{v}{L}, \eta = \frac{L_1}{L}, \quad a_r = \frac{a}{h}, \quad E_R = \frac{E_B}{E_T} \quad (42)$$

Where, η is the ratio of crack location and a_r is the ratio of crack depth.

3. Numerical results

In the numerical examples, the effects of the location of crack, the depth of the crack, the material distributions and different dimensions of the beam on the static deflection of the nanobeams are presented in both the modified couple stress theory and the classical beam theory in s and figures. The beam considered is made of aluminum ($E = 70$ GPa, $\nu = 0.33$), for which the material constants change exponentially as in Eq. (1). The bottom surface of the FGM nanobeam is aluminum. In the numerical calculations, the number of finite elements is taken as $n = 120$.

In the literature, the length scale parameter of an isotropic homogeneous nanobeam has been experimentally evaluated as $l = 17.6 \mu\text{m}$ in (Lam *et al.* 2003). However, there is no available experimental data relevant to the FGM nanobeams in open literature. Therefore, in order to quantitatively study the size effects on the static bending of FGM nanobeams, the values of length

scale parameters are assumed to be equal $l_0 = l_1 = l_2 = l = 15 \mu\text{m}$.

In order to establish the accuracy of the present formulation and the computer program developed by the author, the results obtained from the present study are compared with the available results in the literature. For this purpose, static deflections of a homogeneous cantilever intact beam which is subjected to a point load are calculated for modified couple stress theory and compared with those of Park and Gao (2006) in Fig. 4. It is clearly seen that the curves of Fig. 4 of the present study are very close to those of Fig. 3 of Park and Gao (2006).

In order to investigate the size effect and the material distribution, the maximum dimensionless vertical displacements (at the free end of the beam) of edge cracked FGM nanobeam are presented with different the material distribution (E_R) and the dimensionless material length scale parameter (h/l) for modified couple stress theory and classical beam theory for point load $Q = 1000 \mu\text{N}$, $L/h = 20$, $a_r = 0.8$, $b = h$ and the crack location ratio $\eta = 0.01$ in Table 1.

As seen from Table 1, with the increase in the ratio of the E_R , the displacements decrease, as expected. This is because by increasing the ratio of the E_R , the elasticity modulus of the beam and

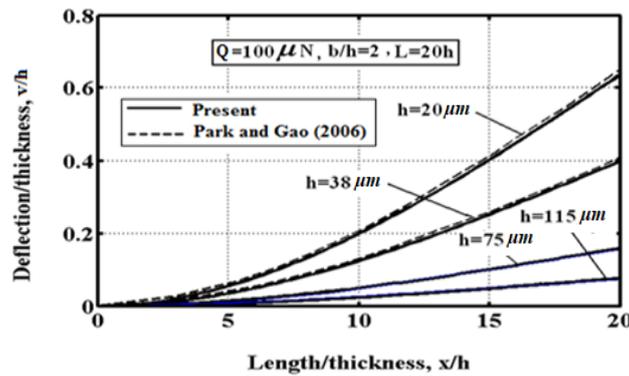


Fig. 4 Deflections of the cantilever beam based on the modified coupled stress theory

Table 1 Maximum dimensionless of vertical displacements edge FGM nanobeam with different the material distribution (E_R) and the dimensionless material length scale parameter (h/l) for point load $Q = 1000 \mu\text{N}$, $L/h = 20$, $a_r = 0.8$, $b = h$ and $\eta = 0.01$

h/l	Modified couple stress theory				Classical beam theory			
	Homogeneous Beam ($E_R = 1$)	$E_R = 0.8$	$E_R = 0.5$	$E_R = 0.1$	Homogeneous Beam ($E_R = 1$)	$E_R = 0.8$	$E_R = 0.5$	$E_R = 0.1$
2	1.2292	1.3085	1.5209	2.9569	1.2851	1.3586	1.5613	2.9789
3	0.5586	0.5925	0.6848	1.3189	0.5712	0.6038	0.6939	1.3240
5	0.2039	0.2158	0.2485	0.4759	0.2056	0.2174	0.2498	0.4766
8	0.0800	0.0847	0.0974	0.1861	0.0803	0.0849	0.0976	0.1862
10	0.0513	0.0542	0.0624	0.1191	0.0514	0.0543	0.0625	0.1192
15	0.0228	0.0241	0.0277	0.0529	0.0228	0.0242	0.0278	0.0530
20	0.0128	0.0136	0.0156	0.0298	0.0129	0.0136	0.0156	0.0298
30	0.0057	0.0060	0.0069	0.0132	0.0057	0.0060	0.0069	0.0132
50	0.0021	0.0022	0.0025	0.0048	0.0021	0.0022	0.0025	0.0048

the bending rigidity increase according to Eq. 1. As a result, the strength of the material increases. It can be noticed that the results predicted by the modified couple stress theory (MCST) are always smaller than those of the classical beam theory (CBT). Further, the difference between the two results is remarkable for thin nanobeams with $h/l < 10$. Also, it is seen from Table 1 that with the decrease in the ratio of the E_R , the difference between the results of the modified couple stress theory and classical beam theory decrease considerably. It shows that an increase in the material length scale parameter (h/l) leads to a decline on effects of size effect and difference between the results of MCST and CBT diminishes for $h/l > 15$.

In Table 2, the maximum dimensionless vertical displacements of the edge cracked FGM nanobeam are presented with different the crack depth ratios (a_r) and the dimensionless material length scale parameter (h/l) for modified couple stress theory and classical beam theory for point load $Q = 5000 \mu\text{N}$, $L/h = 20$, $E_R = 0.5$, $b = h$ and $\eta = 0.01$.

It is seen from Table 2, with the increase in the crack depth, the displacements increase, as expected. This is because by increasing the crack depth ratio, the beam becomes flexible. With the decrease in the crack depth, the difference between the results of the modified couple stress theory and classical beam theory decrease considerably. It shows that an increase in the material length

Table 2 Maximum dimensionless of vertical displacements edge FGM nanobeam with different the crack depth ratios (a_r) and the dimensionless material length scale parameter (h/l) for point load $Q = 5000 \mu\text{N}$, $L/h = 20$, $E_R = 0.5$, $b = h$ and $\eta = 0.01$

h/l	Modified couple stress theory				Classical beam theory			
	Intact	$a_r = 0.1$	$a_r = 0.5$	$a_r = 0.8$	Intact	$a_r = 0.1$	$a_r = 0.5$	$a_r = 0.8$
2	0.1399	0.1461	0.2774	1.5209	0.1803	0.1865	0.3178	1.5613
3	0.0710	0.0728	0.1321	0.6848	0.0801	0.0819	0.1412	0.6939
5	0.0276	0.0282	0.0496	0.2485	0.0288	0.0295	0.0508	0.2498
8	0.0111	0.0113	0.0197	0.0974	0.0113	0.0115	0.0199	0.0976
10	0.0071	0.0073	0.0126	0.0624	0.0072	0.0074	0.0127	0.0625
15	0.0032	0.0033	0.0056	0.0277	0.0032	0.0033	0.0056	0.0278
20	0.0018	0.0018	0.0032	0.0156	0.0018	0.0018	0.0032	0.0156

Table 3 Maximum dimensionless of vertical displacements edge FGM nanobeam with different the crack locations (η) and the dimensionless material length scale parameter (h/l) for point load $Q = 5000 \mu\text{N}$, $L/h = 20$, $E_R = 0.5$, $b = h$ and $a_r = 0.8$

h/l	Modified couple stress theory				Classical beam theory			
	Intact	$\eta = 0.5$	$\eta = 0.3$	$\eta = 0.01$	Intact	$\eta = 0.5$	$\eta = 0.3$	$\eta = 0.01$
2	0.1399	0.4922	0.8303	1.5209	0.1803	0.5326	0.8707	1.5613
3	0.0710	0.2276	0.3779	0.6848	0.0801	0.2367	0.3870	0.6939
5	0.0276	0.0839	0.1380	0.2485	0.0288	0.0852	0.1393	0.2498
8	0.0111	0.0331	0.0542	0.0974	0.0113	0.0333	0.0544	0.0976
10	0.0071	0.0212	0.0347	0.0624	0.0072	0.0213	0.0348	0.0625
15	0.0032	0.0095	0.0155	0.0277	0.0032	0.0095	0.0155	0.0278
20	0.0018	0.0053	0.0087	0.0156	0.0018	0.0053	0.0087	0.0156

scale parameter (h/l) leads to a decline on effects of cracks in difference between the results of MCST and CBT.

In Table 3, the maximum dimensionless vertical displacements of edge cracked FGM nanobeam are presented with different the crack locations (η) and the dimensionless material length scale parameter (h/l) for modified couple stress theory and classical beam theory for $Q = 5000 \mu\text{N}$, $L/h = 20$, $E_R = 0.5$, $b = h$ and $a_r = 0.8$.

It is seen from Table 3 that when the crack locations get closer to the left end of the beam, the displacements increase for cantilever beam. This is because the crack at the left end of the beam has a most severe effect in the cantilever beam. As stated before, the material parameter has a very important role on the static behavior of the edge cracked FGM nanobeams, and it should be considered in the static and dynamic analysis of nanobeams. Also, it is believed that the tabulated results will be a reference with which other researchers can compare their results.

Also it is seen from Table 3 that increase in the material length scale parameter (h/l), the difference between results of the MCST and CBT decrease considerably for different ratio of the crack locations (η). It is observed from tables that for the higher ratio of h/l , the classical cracked-beam theory can be used for the cracked problems of the nanobeams.

In order to investigate the effect of crack depth ratio (a_r) on the results of MCST and CBT, the maximum dimensionless vertical displacements of edge cracked FGM nanobeam are presented with different crack depth ratio in both MCST and CBT for $Q = 1000 \mu\text{N}$, $L/h = 20$, $E_R = 0.1$, $h/l = 3$, $b = h$ and $\eta = 0.01$ in Fig. 5.

As seen from Fig. 5 that with increase in the crack depth ratios (a_r), the difference between results of the MCST and CBT increases. In small values of the crack depth ratios, the difference between results of the MCST and CBT decrease considerably. It is observed from Fig. 5 that for small values of the crack depth ratios, the classical cracked-beam theory can be used for the cracked problems of the nanobeams.

In Fig. 6, the maximum dimensionless vertical displacements of the edge cracked FGM nanobeam are presented with different the crack depth ratios (a_r) and the slenderness ratios (L/h)

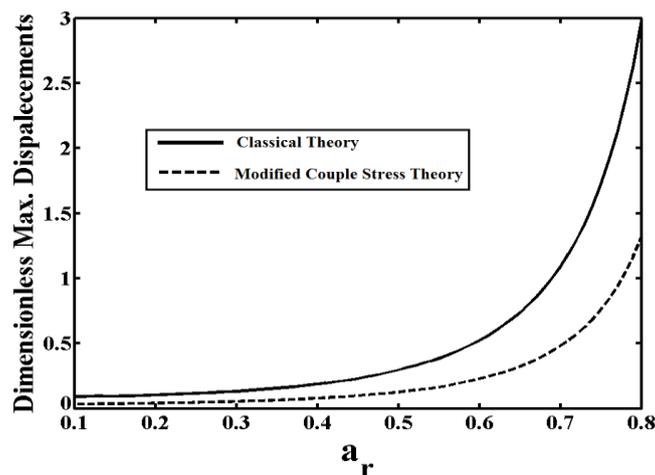


Fig. 5 Maximum dimensionless vertical displacements of edge cracked FGM nanobeam with different the crack depth ratios (a_r) in both MCST and CBT for $Q = 1000 \mu\text{N}$, $L/h = 20$, $E_R = 0.1$, $h/l = 3$, $b=h$ and $\eta = 0.01$

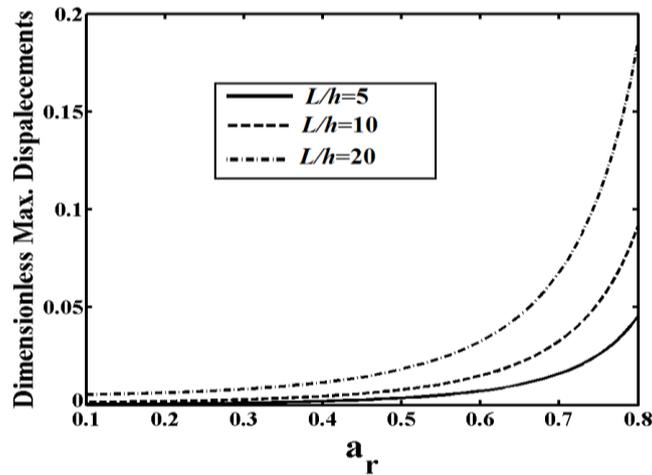


Fig. 6 Maximum dimensionless vertical displacements of edge cracked FGM nanobeam with different the crack depth ratios (a_r) and the slenderness ratios (L/h) for $Q = 1000 \mu\text{N}$, $h/l = 8$, $E_R = 0.1$, $b = h$ and $\eta = 0.01$

on modified couple stress theory for $Q = 1000 \mu\text{N}$, $h/l = 8$, $E_R = 0.1$, $b = h$ and $\eta = 0.01$.

It is seen from Fig. 6 that with increase in the ratio of L/h , the difference among the crack depth ratios increase significantly. This is because by increasing the ratio of the L/h , the beam gets more flexible and makes more sensitive the cracks. It shows that the geometry properties have a very important role on the static behavior of the edge cracked nanobeams.

In Fig. 7, the effects the crack locations (η) on the results of MCST and CBT are plotted for $Q = 1000 \mu\text{N}$, $L/h = 20$, $E_R = 0.1$, $h/l = 3$, $b = h$ and $a_r = 0.6$.

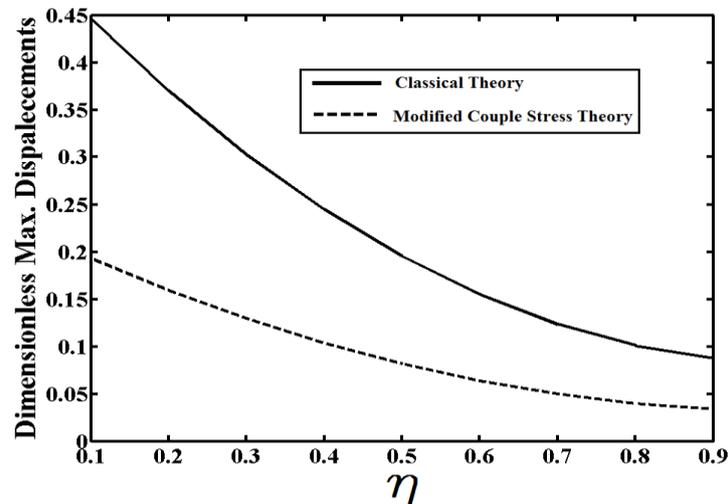


Fig. 7 Maximum dimensionless vertical displacements of edge cracked FGM nanobeam with different the crack locations (η) in both MCST and CBT for $Q = 1000 \mu\text{N}$, $L/h = 20$, $E_R = 0.1$, $h/l = 3$, $b = h$ and $\eta = 0.01$

It is seen from Fig. 7 that with decrease in the crack locations (η), in other words, when the crack locations get closer to the left end of the beam, the difference between results of the MCST and CBT increases considerably. This is because the crack at the left end of the beam has a most severe effect in the cantilever beam. Therefore, the beam gets more flexible and makes more sensitive the cracks. The reasons for large difference between MCST and CBT in small values of the crack locations is by using the classical cracked-beam theory for the cracked nanobeams based on the modified couple stress theory (MCST). It is observed from Fig. 7 that in high values of the crack locations, in other words, when the crack locations get closer to the free end of the beam, the difference between results of the MCST and CBT decreases because the effect of the crack on the beam is low when the crack gets closer to the free end.

In Fig. 8, the maximum dimensionless vertical displacements of the edge cracked FGM nanobeam are presented with different the crack locations (η) and the slenderness ratios (L/h) on modified couple stress theory for $Q = 1000 \mu\text{N}$, $h/l = 8$, $E_R = 0.1$, $b = h$ and $a_r = 0.8$.

It is seen from Fig. 8 that with increase in the ratio of L/h , the difference among the crack depth ratios increase significantly. It is mentioned before that, because by increasing the ratio of the L/h , the beam gets more flexible and makes more sensitive the cracks. Also it seen from Fig. 8 that, with increase in the crack locations (η), the difference among the results of the L/h decreases considerably as like the results of the Fig. 7.

In order to investigate relationship between the effect of crack depth ratio (a_r) and the material distribution, the maximum dimensionless vertical displacements (at the free end of the beam) of edge cracked FGM nanobeam are presented with different ratio of E_R and a_r on modified couple stress theory for $Q = 1000 \mu\text{N}$, $L/h = 20$, $h/l = 8$, $b = h$ and $\eta = 0.01$ in Fig. 9.

It is seen from Fig. 9 that increases in the Young's modulus ratio E_R causes decrease in the displacements. When the Young's modulus ratio E_R increases, the elasticity modulus of the beam increases according to Eq. (1). Another result of the Fig. 9 that with the increase in Young's modulus ratio E_R , the differences of the crack depth ratio a_r decrease seriously. This is because, an increase in the Young's modulus ratio E_R leads to an increase in the elasticity modulus and the

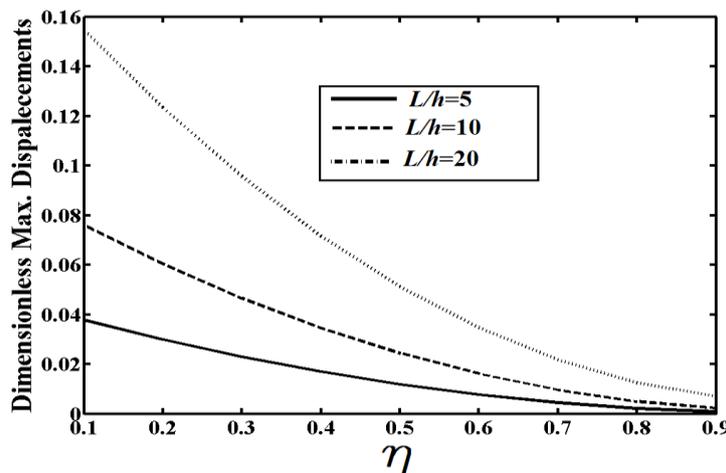


Fig. 8 Maximum dimensionless vertical displacements of edge cracked FGM nanobeam with different the crack locations (η) and the slenderness ratios (L/h) for $Q = 1000 \mu\text{N}$, $h/l = 8$, $E_R = 0.1$, $b = h$ and $a_r = 0.8$

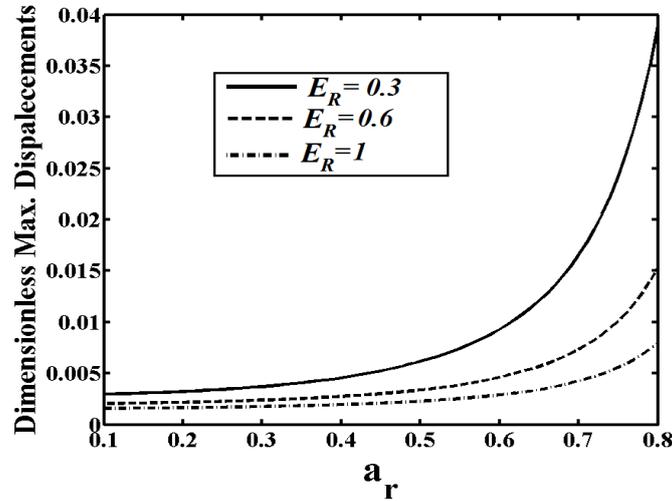


Fig. 9 Maximum dimensionless vertical displacements of edge cracked FGM nanobeam with different the crack depth ratios (a_r) and the material distribution ratio (E_R) for $Q = 1000 \mu\text{N}$, $L/h = 20$, $h/l = 8$, $b = h$ and $\eta = 0.01$

bending rigidity. As a result, the strength of the material increases. Hence, the beam becomes more strength and the effect of the crack on the FGM nanobeam decreases. It shows that with the suitable choice of E_R , the negative effects of the crack can be reduced.

In Fig. 10, the maximum dimensionless vertical displacements of the edge cracked FGM nanobeam are presented with different the crack locations (η) and the material distribution on modified couple stress theory for $Q = 1000 \mu\text{N}$, $L/h = 20$, $h/l = 8$, $b = h$ and $a_r = 0.8$.

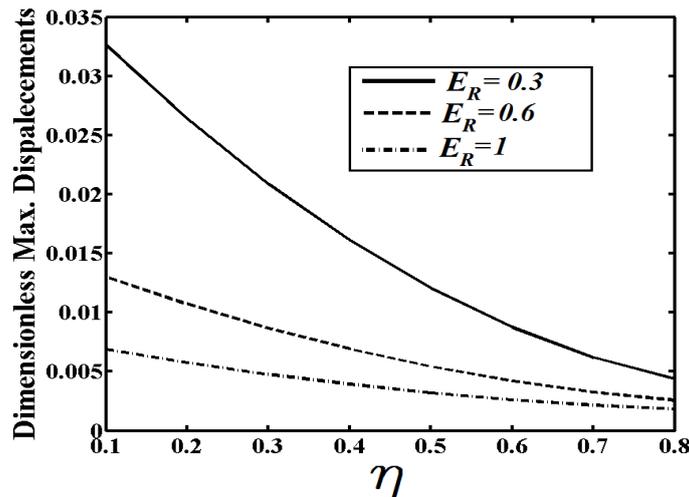


Fig 10 Maximum dimensionless vertical displacements of edge cracked FGM nanobeam with different the crack locations (η) and the material distribution ratio (E_R) for $Q = 1000 \mu\text{N}$, $L/h = 20$, $h/l = 8$, $b = h$ and $a_r = 0.8$

It is observed from Fig. 10 that with the increase in Young's modulus ratio E_R , the differences of the crack locations decrease seriously. It is observed from results that the distribution of the material plays an important role on the mechanical behaviour of the FGM nanobeam. It shows that FGM is very effective for reducing the negative influence of the cracks. Also, it is observed from Fig. 10 that in high values of the crack locations, in other words, when the crack locations get closer to the free end of the beam, the difference between results of the ratio E_R decreases because the effects of the crack on the beam diminish when the crack gets closer to the free end.

Figs. 11, 12 and 13 display the effect of the material distribution, the crack depth and the crack location on the deflected shape of the beams for $Q = 1000 \mu\text{N}$, $L/h = 20$, $h/l = 8$, $b = h$.

It is observed from Fig. 11 that with the increase in the ratio of the E_R , the displacements

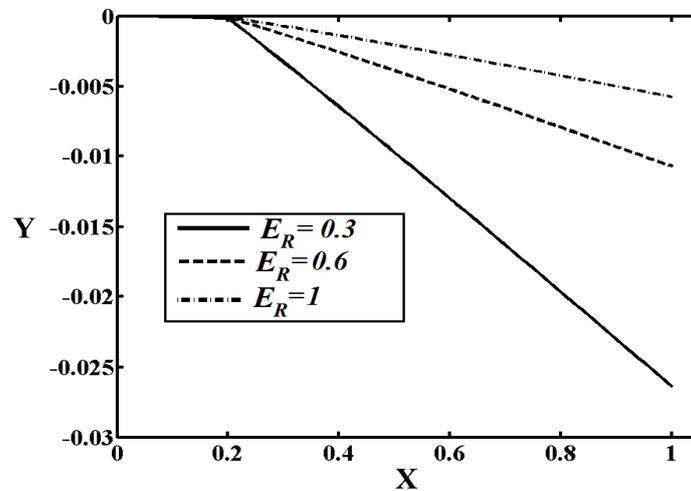


Fig. 11 The effect of the material distribution ratio (E_R) on the deflected shape of the FGM nanobeam for $Q = 1000 \mu\text{N}$, $L/h = 20$, $h/l = 8$, $b = h$, $\eta = 0.2$ and $a_r = 0.8$

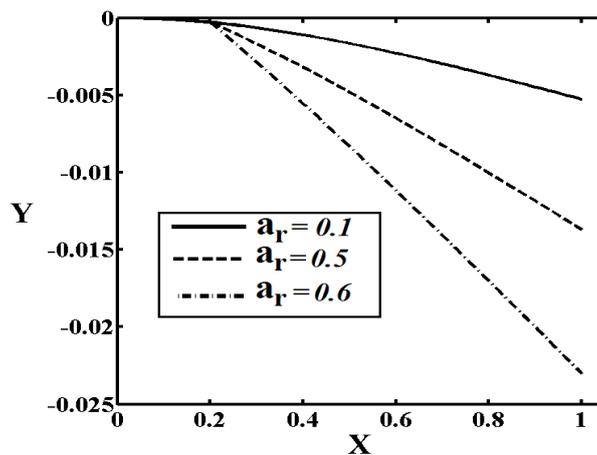


Fig. 12 The effect of the crack depth ratios (a_r) on the deflected shape of the FGM nanobeam for $Q = 1000 \mu\text{N}$, $L/h = 20$, $h/l = 8$, $b = h$, $\eta = 0.2$ and $E_R = 0.1$.

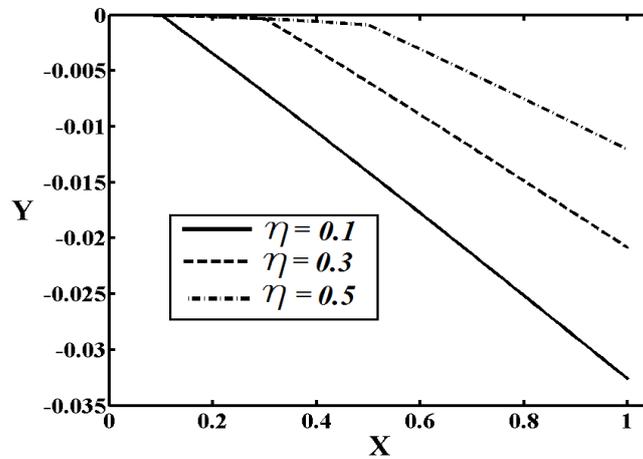


Fig. 13 The effect of the crack locations (η) on the deflected shape of the FGM nanobeam for $Q = 1000$
 μN , $L/h = 20$, $h/l = 8$, $b = h$, $a_r = 0.8$ and $E_R = 0.1$

decrease, as expected. It is stated before that because of increasing the ratio of the E_R , the elasticity modulus of the beam and the bending rigidity increase according to Eq. (1). As a result, the strength of the material increases. Clearly, by having a suitable material distribution of the FGM, the negative effect of the crack can be reduced. Evidently, Young's modulus ratio E_R plays a very important role in the mechanical response of the FGM nanobeam.

From Fig. 12, it is seen that with the increase in the crack depth ratio, the displacement increases, as expected. This is due to the fact that increasing the crack depth ratio results in the increase of the flexibility of the FGM nanobeam. The results indicate that the cracks are very sensitive to the change in the mechanical behaviour of the FGM nanobeam.

It is clearly seen from Fig. 13 that when the crack location gets closer to the left end (fixed end), the displacement of the FGM nanobeam increases, due to the fact that the crack at the fixed end has a most severe effect on the beam. So, when the crack location gets closer to the fixed end, the beam gets more flexible. It was mentioned previously that as the crack gets closer to the left end, the strength of the beam decreases. As can be seen, the crack location plays an important role in the static response of the FGM nanobeams.

4. Conclusions

This paper examines static bending of an edge cracked FGM cantilever nanobeam subjected to transversal point load at the free end based on the modified couple stress theory by using finite element method within the Euler-Bernoulli beam theory. Material properties of the beam change in the height direction according to exponential distributions. The cracked beam is modelled using a proper modification of the classical cracked-beam theory consisting of two sub-beams connected through a massless elastic rotational spring. The elastic deflections of the edge cracked nano beams are calculated and discussed for different crack positions, different lengths of the beam, different length scale parameter, different crack depths, and different material distributions. In order to establish the accuracy of the present formulation and results, the deflections are obtained,

and compared with the published results available in the literature. Good agreement is observed. The primary purpose of this study is to fill this gap for FGM nanobeams.

The following conclusions are reached from the obtained results:

- (1) With the suitable choice of material distribution, the negative effects of the crack can be reduced.
- (2) FGM is very effective for reducing the negative influence of the cracks.
- (3) The crack location and the crack depth play an important role in the static response of the FGM nanobeams.
- (4) It is found that the deflections of the FGM nanobeam by the classical beam theory are always larger than those by the modified couple stress theory.
- (5) The geometry properties and material parameter have a very important role on the static behavior of the edge cracked FGM nanobeams.
- (6) With increase in the material length scale parameter (h/l), the difference between results of the MCST and CBT decrease considerably for different ratio of the crack locations (η) and the crack depth ratio (a_r).
- (7) With increase in the ratio of h/l , the effects of the cracks on the static responses of the nano beams are decrease significantly. With increase in the ratio of h/l , the difference among the crack depth ratios diminishes. After a certain value of the ratio of h/l and L/h , the classical cracked-beam model can be used for the cracked problems of the nanobeams.
- (8) With increase in the crack locations (η), the difference among the results of the L/h decreases considerably.
- (9) Young's modulus ratio E_R plays a very important role in the mechanical response of the FGM nanobeam.
- (10) For the smaller ratio of L/h , the modified couple stress theory must be used instead of the classical beam theory.
- (11) It is believed that the tabulated results will be a reference with which other researchers can compare their results.

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Appendix

The interpolation functions for axial degrees of freedom are

$$\varphi^{(U)}(X) = \left[\varphi_1^{(U)}(X) \ \varphi_2^{(U)}(X) \right]^T, \quad (\text{A1})$$

Where

$$\varphi_1^{(U)}(X) = \left(-\frac{X}{L_e} + 1 \right), \quad (\text{A2})$$

$$\varphi_2^{(U)}(X) = \left(\frac{X}{L_e} \right) \quad (\text{A3})$$

The interpolation functions for transverse degrees of freedom are

$$\varphi^{(V)}(X) = \left[\varphi_1^{(V)}(X) \ \varphi_2^{(V)}(X) \ \varphi_3^{(V)}(X) \ \varphi_4^{(V)}(X) \right]^T, \quad (\text{A4})$$

Where

$$\varphi_1^{(V)}(X) = \left(1 - \frac{3X^2}{L_e^2} + \frac{2X^3}{L_e^3} \right) \quad (\text{A5})$$

$$\varphi_2^{(V)}(X) = \left(-X + \frac{2X^2}{L_e} - \frac{X^3}{L_e^3} \right) \quad (\text{A6})$$

$$\varphi_3^{(V)}(X) = \left(\frac{3X^2}{L_e^2} - \frac{2X^3}{L_e^3} \right) \quad (\text{A7})$$

$$\varphi_4^{(V)}(X) = \left(\frac{X^2}{L_e} - \frac{2X^3}{L_e^3} \right) \quad (\text{A8})$$

Where L_e indicates the length of the finite beam element.