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Vibration analysis of carbon nanotubes with multiple cracks in thermal environment

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Abstract. In this study, the thermal loading effect on free vibration characteristics of carbon nanotubes (CNTs) with multiple cracks is studied. Various boundary conditions for nanotube are taken in to account. In order to take the small scale effect, the nonlocal elasticity of Eringen is employed in the framework of Euler-Bernoulli beam theory. This theory states that the stress at a reference point is a function of strains at all points in the continuum. A cracked nanotube is assumed to be consisted of two segments that are connected by a rotational spring which is located in the position of the cracked section. Hamilton's principle is used to achieve the governing equations. Influences of the nonlocal parameter, crack severity, temperature change and the number of cracks on the system frequencies are investigated. Also, it is found that at room or lower temperature the natural frequency for CNT decreases as the value of temperature change increases, while at temperature higher than room temperature the natural frequency of CNT increases as the value of temperature change increases. Various boundary conditions have been applied to the nanotube.

Keywords: nonlocal elasticity theory; vibration; thermal effect; crack; carbon nanotube

1. Introduction

Nanostructures have singular properties and are useful in many disciplines so recently, researchers are interested to study their several analyses and examine different effects on this analysis. One of this analysis is vibration that investigators study the influence of various multiphysics phenomena on the vibration behavior of nanostructures such as nanorods (Murmu *et al*. 2014), nanobeams (Kiani 2012), nanoplates (Murmu *et al*. 2013), etc. Most of these studies are related to the usage of nanostructures, which are commonly used in nanoelectromechanical systems (NEMS) devices (e.g., resonators) with different boundary and loading conditions. Today, investigating the mechanical, physical and thermal properties have been become very important in nanoengineering practice (Haghshenas and Arani 2013). There are three basic methods for nanostructures analysis: experimental analysis (Meyer *et al*. 2007), molecular dynamic simulation (Park *et al*. 2005) and the continuum mechanics approach (Eringen 1972, 1983, Reddy and Pang

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2008). Experimental studies of nanostructures are very important for determining their physical properties. But, this approach is very expensive, and also, very small dimensions of nanostructures and weak control of experimental parameters makes direct measurement of properties difficult. On the other hand, Molecular dynamics simulation method was developed for the dynamic behavior of nanostructures. But, this method is only support the nanostructures with a small number of atoms and molecules whereas for large size of nanoscale system, it is time-consuming approach and have prohibitive computationally.

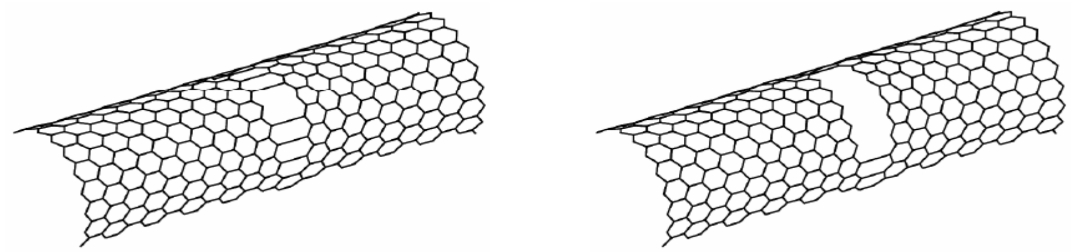
All this leads that the continuum-based theory is the best approach. Using non-local theories for carbon nanotubes analyze, help us to understand the mechanical behavior of materials at small scale. This kind of theory was included information about the forces between atoms and internal length (small scale effect), in which the structural relation for material parameters have been defined. The main difference between the classical theory and the theory of elasticity is in the definition of nonlocal stress. Nonlocal theory is first considered by Eringen, this theory expresses that the stress at a reference point is considered to be a functional of the strain field at every point in the body (Eringen 1972, 1983). The nonlocal elasticity theory has been used to investigate wave propagation, dislocation mechanics, crack problems and elastic waves and vibration analysis (Ebrahimi and Salari 2015a, b, 2016, Ebrahimi *et al*. 2015a, 2016a, Ebrahimi and Nasirzadeh 2015, Ebrahimi and Barati 2016a-f, Ebrahimi and Hosseini 2016a-c, Wang 2005).

In Eltaher *et al*. (2016a) the review on bending, buckling, vibrations, and wave propagation of nanobeams modeled according to the nonlocal elasticity theory of Eringen was investigated. Karličić *et al*. (2015b) investigate on a theoretical study of the free longitudinal vibration of a nonlocal viscoelastic double-nanorod system. In their paper D'Alembert's principle is applied to derive the governing equations of motion and the solutions of these partial differential equations are obtained by using the classical Bernoulli-Fourier method. The free flexural vibration and buckling of SWCNT under compressive axial loading based on the classical Euler-Bernoulli or Reddy beam theory are study by Karličić *et al*. (2015b), too. Also, in other paper they modeled the nonlinear model of a SWCNT based on the nonlocal Euler-Bernoulli beam theory, Maxwell's equations and von Karman nonlinear strain-displacements relation. They study the nonlocal parameter, magnetic field effects and stiffness coefficient of the viscoelastic medium that have significant effects on vibration and stability behavior of nanobeam (Karličić *et al*. 2017).

Also, in recent years, researchers are interested to study the influence of different physical effects on dynamic behavior of one-dimensional nanostructures like carbon nanotube (CNT) and single-walled carbon nanotube (SWCNT) by using nonlocal theory. Murmu and Pradhan (2009) have studied the thermal and nonlocal effects on the free vibration of CNT embedded in an elastic medium. In Murmu and Pradhan paper (2010) nonlocal beam model is applied to the buckling analysis of SWCNT with effect of temperature change and surrounding elastic medium. Also, the stability of CNT in an elastic medium under the influence of temperature change has been showed. In both paper they used Euler–Bernoulli beam theory. Benzair *et al*. (2008) reformulated the classical Timoshenko beam theory by using the nonlocal elasticity. In addition, they introduced thermal effects by constitutive relation for vibration analysis of CNT. In (Ke and Wang 2012) Hamilton principle and nonlocal elasticity have been used to derive the governing equations of motion. Furthermore, they investigated the thermal effect on vibration of SWCNTs by using Timoshenko beam theory. Wang *et al*. (2008) have studied the thermal effects on the vibration and instability of CNTs conveying fluid based on thermal elasticity mechanics. In their paper an elastic Bernoulli–Euler beam model is developed for the vibration and buckling instability of SWNTs conveying fluid, and the effect of temperature change on the properties of buckling instability is examined, too. Zhang *et al*. (2008) have studied the thermal effect on the vibration of double walled carbon nanotubes using thermal elasticity mechanics. Most recently Ebrahimi and Barati (2016g-v, 2017a, b) and Ebrahimi *et al*. (2017) explored thermal and hygro-thermal effects on nonlocal behavior of FG nanobeams and nanoplates.

The study of the vibration behavior of cracked CNTs are important to recognize the mechanical behavior of these nanostructures. The influence of these cracks place and cracks severity are demonstrated in the nanostructure's stiffness. The flexural vibrations of cracked micro- and nanobeams are studied by Loya *et al*. (2009), their model is based on the theory of nonlocal elasticity applied to Euler–Bernoulli beams. In their study the nanobeam is separated into two parts and the crack is simulated by a rotational spring. The crack will not only change the stiffness of nanobeam but also alter the damping properties, and all these changes will affect the vibration characteristic. In addition, the researchers have examined the effect of crack hardness, the nonlocal parameter and boundary conditions on natural frequencies of the cracked nanobeam. Hsu *et al*. (2011) studied the longitudinal frequency of a cracked nanobeam for two type of boundary conditions, they use the nonlocal elasticity theory and investigated different effects such as the crack parameter, crack location, and nonlocal parameter on the longitudinal frequency of the cracked nanobeam. Torabi and Dastgerdi (2012) have investigated the free vibration of a cracked nanobeam modeled via nonlocal elasticity and Timoshenko beam theory, where the cracked nanobeam is represented by two segments connected by a rotational spring. They analyzed the effects of crack position, crack ratio and the nonlocal parameter on the vibration mode and frequency parameter. Yang and Chen (2008) provided a theoretical investigation the free vibration and elastic buckling of beams made of functionally graded materials (FGMs) containing open edge cracks by using Bernoulli–Euler beam theory and the rotational spring model. The bending vibrations of a cracked nanobeam with surface effects were studied by Hasheminejad *et al*. (2011). The vibration behavior of a nanobeam with multiple cracks for different boundary conditions were studied by Roostai and Haghpanahi (2014). The influence of changing the number of cracks on dimensionless frequencies for changing the boundary conditions was shown too. We try to give some comparative of present paper with results of this article. The free transverse vibration of a size-dependent cracked functionally graded (FG) in framework of Timoshenko nanobeam theory was study by Ghadiri *et al*. (2016) and they study the different influence of surface effects on vibration behavior for this FG nanobeam, too. In other paper Karličić *et al*. (2015b) study the thermal and magnetic effects on the free vibration of a cracked nanobeam embedded in an elastic medium they chose an Euler–Bernoulli beam theory based on the nonlocal elasticity, and the result of this article is near to the result of present paper. Rahmani *et al*. (2015) investigate the torsional vibration of cracked nanobeam that based on a nonlocal elasticity theory and also, study different parameters effects on vibration for various boundary conditions.

The study of the vibration behavior of cracked CNTs is of great theoretical and practical interest for better understanding of the mechanical response of nanostructures. Also studies on the different effect on the vibration behavior are important for researcher. But, most of the investigations of vibration problems of multiple cracked CNTs have not considered the significant effects such as thermal effect. Thus, the investigations of thermal effect on them should be interesting and necessary. So, in this study the influence of temperature on vibration behavior of nanotube of different boundary conditions with multiple cracks were investigated and it is the main motivation of the presented paper. To provide an analytical model and analyze the vibrational behavior of a cracked nanotube by taking into account thermal effects have been studied. The Hamilton's principle was utilized to deriving to the governing equations of CNT. Therefore, it is



(a) (b)

Fig. 1 Evolution of the crack in the nanotube (Belytschko 2002)

desirable to propose an analytical method for solving the title problem. Natural frequencies are extracted for cracked nanotubes, and also the effect of some different parameters such as the crack positions, crack severities, and values of the nonlocal length parameter in typical boundary conditions are investigated.

2. Problem formulation

2.1 Free vibration of an intact nanotube

A schematic diagram of nanotube with multiple cracks exposed to axial force is portrayed in Fig. 2. The nanotube has n cracks which located at X_i (i = 1,..., n).

The main practical applications of the given calculations in this paper are analytical method that we extracted below formulation by using nonlocal theory that was proposed by Eringen after that guess a responses and use analytical method to solve them.

In this subsection, the fundamental constitutive relation of the nonlocal elasticity and thermoelasticity theory has been provided. The basic equations for a linear homogenous nonlocal elastic body neglecting the body forces for present nanotube can be expressed as follows

$$\sigma_{ij}(x) = \int \alpha(|x - x'|.\tau) C_{ijkl}\, \epsilon_{kl}(x') dV(x'). \tag{1a}$$

for all $x \in V$

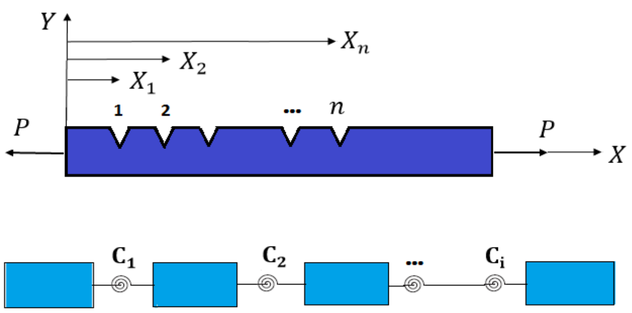


Fig. 2 Nanotube with multiple cracks subjected to axial force

$$\sigma_{ij,j} = 0. \tag{1b}$$

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{1c}$$

Where C_{ijkl} is the elastic modulus tensor for classical isotropic elasticity; σ_{ij}, ϵ_{ij}, u_i are respectively, the stress tensor, the strain tensor and the displacement vector. The main function $\alpha(|x - x'|)$ is nonlocal modulus, which is attenuation function incorporating into constitutive equations the nonlocal effects at the reference point x produced by local strain at the source x'. The above absolute value of the difference $|x - x'|$ represents the Euclidean form. The term $\tau = \frac{e_0 a}{l}$ is a material constant, where l is the external characteristic length (crack length and wave length), a describes the internal characteristic length (lattice parameter, granular size and distance between C-C bonds) and e_0 is a calibration constant appropriate to each material such that the relations of the nonlocal elasticity model could provide satisfactory approximation to the atomic dispersion curves of the plane waves with those obtained from the atomistic lattice dynamics (Murmu and Adhikari 2010). Based on Eringen (1972), constitutive relations in the one-dimensional differential form of nonlocal elastic equation for the case can be written as

$$\sigma_{xx} - \mu \frac{d^2 \sigma_{xx}}{dx^2} = E\epsilon_{xx}. \tag{2}$$

Where E and G are respectively, the elastic modulus and the shear modulus of the beam. $\mu = (e_0 a)^2$ is the nonlocal parameter (length scales); σ_{xx} and σ_{xz} are the normal and the shear nonlocal stresses, respectively, and $\epsilon_{xx} = u - z\frac{\partial^2 \omega}{\partial x^2}$ is the axial deformation. Nanomaterials such as CNTs, ZnO nanotubes and other one-dimensional structures are modeled as nanobeams and nanorods by using the nonlocal theory, where internal characteristic lengths $e_0 a$ are often assumed to be in the range 0–2 nm. When $e_0 a = 0$, the nonlocal constitutive relation be converted to the classical constitutive relation of the elastic body. A conservative estimate of the scale coefficient $e_0 a$ < 2.0 nm for a SWCNT has been proposed by Wang (2005) and the parameter $e_0$was proposed as 0.39 by Eringen (1972). Zhang *et al*. have proposed the nonlocal thermoelastic constitutive relation model and a combination of nonlocal elasticity and classical thermoelasticity theory studied by Murmu and Pradhan (2009, 2010). Therefore, for one-dimensional nonlocal viscoelastic solids, constitutive relations are given by

$$\sigma_{xx} - \mu \frac{d^2 \sigma_{xx}}{dx^2} = E\left(\epsilon_{xx} - \frac{\alpha_x \theta}{1 - 2\nu}\right). \tag{3}$$

Where α_x; ν are denoting the coefficient of thermal expansion in the direction of the x axis and the Poisson ratio, respectively, and θ is the change in temperature. If $\theta = 0$ i.e., that is mean equation haven't influence of change in temperature, the equation has been returned to the constitutive relation for nonlocal elasticity.

According to Newton's second law, the dynamic force equilibrium conditions of these forces are given in the following differential equations of the system, as in (Kozić *et al*. 2014), which gives

$$\frac{\partial F}{\partial X} + \mathrm{N}\frac{\partial^2 \omega}{\partial X^2} = \rho A \frac{\partial^2 \omega}{\partial t^2}. \tag{4a}$$

$$\frac{\partial N}{\partial X} = \rho A \frac{\partial^2 u}{\partial t^2}. \tag{4b}$$

$$F_T = \frac{\partial M_f}{\partial X}. \tag{4c}$$

The resultant bending moment (the reaction induced in a structural element when an external force or moment is applied to the element) for a nanotube to be considered by

$$M_f = \int_0^A z\sigma_{xx}\, dA. \tag{5a}$$

And axial force can be displayed by

$$N = \int_0^A \sigma_{xx}\, dA. \tag{5b}$$

By using the nonlocal constitutive relation from (3) with cases (4) and (5) and assuming that the axial displacement $u = 0$, the equations are obtained as follows

$$M_f = \mu\left[-N\frac{\partial^2 \omega}{\partial X^2} + \rho A\frac{\partial^2 \omega}{\partial t^2}\right] - EI\frac{\partial^2 \omega}{\partial X^2}. \tag{6a}$$

$$N = -EA\frac{\alpha_x \theta}{1-2\nu}. \tag{6b}$$

Substitution of Eqs. (6a), (6b) and (4c) into (4a) leads to the following equation that is the equation of motion the nanotube, this case can be showed by

$$\mu\left[\rho A\frac{\partial^4 \omega}{\partial X^2 \partial t^2} - N\frac{\partial^4 \omega}{\partial X^4}\right] - EI\frac{\partial^4 \omega}{\partial X^4} = -N\frac{\partial^2 \omega}{\partial X^2} + \rho A\frac{\partial^2 \omega}{\partial X^2}. \tag{7}$$

Displacement all the parameter in above equation yields the differential equation for modal displacements and it is given in the following

$$EI\frac{\partial^4 \omega}{\partial X^4} + (e_0 a)^2\left[EA\frac{\alpha_x \theta}{1-2\nu}\frac{\partial^4 \omega}{\partial X^4} - \rho A\frac{\partial^4 \omega}{\partial X^2 \partial t^2}\right] - N\frac{\partial^2 \omega}{\partial X^2} + \rho A\frac{\partial^2 \omega}{\partial X^2} = 0. \tag{8}$$

For free vibration, we can obtain the equation by assumed transverse displacement in this form $\omega(X.t) = Y(X)e^{i\omega_c t}$, Where ω_c is the angular frequency. The dimensionless parameters are given as follows

$$x = \frac{X}{l}; \quad y = \frac{Y}{l}; \quad \tau = \frac{e_0 a}{l}; \quad q = EA\frac{\alpha_x \theta}{1-2\nu}\frac{l^2}{EI}; \quad \omega = \omega_c l^2\sqrt{\frac{\rho}{EI}}. \tag{9}$$

In the following, the partial differential equation of motion (Eq. (9)) for this nanobeam can be simplified in this form

$$\frac{\partial^4 y}{\partial x^4}(1+\tau^2 q)+\frac{\partial^2 y}{\partial x^2}(\tau^2\omega^2-q)-\omega^2 y=0. \tag{10}$$

The exponential form of the solution (Eq. (10))

$$y(x)=Ae^{isx}. \tag{11}$$

When inserting Eq. (10) into Eq. (11) can have

$$s^4(1+\tau^2 q)-s^2(\tau^2\omega^2-q)-\omega^2=0. \tag{12}$$

The solution is

$$s_{1.2}=\pm i\sqrt{\frac{P_1+P_3-P_2}{P_4}}=\pm i\alpha;\qquad s_{3.4}=\pm i\sqrt{\frac{P_2+P_3-P_1}{P_4}}=\pm\beta \tag{13}$$

where

$$P_1=q;\quad P_2=\varepsilon^2\omega^2;\quad P_3=\sqrt{(q+\varepsilon^2\omega^2)^2+4\omega^2};\quad P_4=2(1+\varepsilon^2 q) \tag{14}$$

In this case, Eq. (11) outputs

$$y(x)=c_1e^{\alpha x}+c_2e^{-\alpha x}+c_3e^{\beta x}+c_4e^{-\beta x}. \tag{15}$$

Consider

$$S_1=e^{\alpha x};\ S_2=e^{-\alpha x};\ S_3=e^{\beta x};\ S_4=e^{-\beta x}. \tag{16}$$

In order to comfort the analysis, the linearly independent fundamental solutions are denoted by $S_j(x)\ \ (i=1.2.3.4)$; which satisfy the following normalization condition at the origin of coordinate system (Li 2001).

$$\begin{bmatrix}\bar{S}_1(0) & \bar{S}_1'(0) & \bar{S}_1''(0) & \bar{S}_1'''(0)\\ \bar{S}_2(0) & \bar{S}_2'(0) & \bar{S}_2''(0) & \bar{S}_2'''(0)\\ \bar{S}_3(0) & \bar{S}_3'(0) & \bar{S}_3''(0) & \bar{S}_3'''(0)\\ \bar{S}_4(0) & \bar{S}_4'(0) & \bar{S}_4''(0) & \bar{S}_4'''(0)\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=I_4. \tag{17}$$

Easily we can find the $\bar{S}_j(x)$ by

$$\begin{Bmatrix}\bar{S}_1(x)\\ \bar{S}_2(x)\\ \bar{S}_3(x)\\ \bar{S}_4(x)\end{Bmatrix}=\begin{bmatrix}\bar{S}_1(0) & \bar{S}_1'(0) & \bar{S}_1''(0) & \bar{S}_1'''(0)\\ \bar{S}_2(0) & \bar{S}_2'(0) & \bar{S}_2''(0) & \bar{S}_2'''(0)\\ \bar{S}_3(0) & \bar{S}_3'(0) & \bar{S}_3''(0) & \bar{S}_3'''(0)\\ \bar{S}_4(0) & \bar{S}_4'(0) & \bar{S}_4''(0) & \bar{S}_4'''(0)\end{bmatrix}^{-1}\begin{Bmatrix}S_1(x)\\ S_2(x)\\ S_3(x)\\ S_4(x)\end{Bmatrix}. \tag{18}$$

The primes in Eqs. (17) and (18) indicate the differentiation with respect to x. $\bar{S}_j(x)$ obtain in the form

$$\bar{S}_1(x) = \delta(\beta^2 \cosh \alpha x + \alpha^2 \cos \beta x). \tag{19a}$$

$$\bar{S}_2(x) = \delta\left(\frac{\beta^2}{\alpha} \sinh \alpha x + \frac{\alpha^2}{\beta} \sin \beta x\right). \tag{19b}$$

$$\bar{S}_3(x) = \delta(\cosh \alpha x - \cos \beta x). \tag{19c}$$

$$\bar{S}_4(x) = \delta\left(\frac{1}{\alpha} \sinh \alpha x - \frac{1}{\beta} \sin \beta x\right). \tag{19d}$$

$$\delta = \frac{1}{\alpha^2 + \beta^2}. \tag{19e}$$

The general solution (Eq. (15)), at $x = 0$ as a function of boundary conditions can be written in the form

$$y(x) = y(0)\bar{S}_1(x) + y^{'}(0)\bar{S}_2(x) + y^{''}(0)\bar{S}_3(x) + y^{'''}(0)\bar{S}_4(x). \tag{20}$$

2.2 Free vibration of nanotube with multiple cracks

The influence of the crack in the nanotube was subsided by dividing the classical cracked nanotube element into two segments connected by a rotational spring located at the cracked section. It is assumed that the crack is open. For the first segment of the nanotube could have the shape function as follow

$$y_1(x) = y_1(0)\bar{S}_1(x) + y_1^{'}(0)\bar{S}_2(x) + y_1^{''}(0)\bar{S}_3(x) + y_1^{'''}(0)\bar{S}_4(x). \tag{21}$$

The continuity conditions at the crack position between two adjacent sections as follows

$$y_i(x_i) = y_{i+1}(x_i). \tag{22a}$$

$$y_i^{''}(x_i) = y_{i+1}^{''}(x_i). \tag{22b}$$

$$y_{i+1}^{'}(x_i) - y_i^{'}(x_i) = C_i y_i^{''}(x_i). \tag{22c}$$

$$y_i^{'''}(x_i) + \psi y_i^{'}(x_i) = y_{i+1}^{'''}(x_i) + \psi y_{i+1}^{'}(x_i). \tag{22d}$$

$$\psi = \varepsilon^2 \omega^2. \tag{22e}$$

Where x_i the dimensionless position of the ith is crack and C_i is the dimensionless flexibility of the rotational spring (crack severity). For the 2nd segment of the nanotube can have the shape function as

$$y_2(x) = y_1(x) + C_1 y_1^{''}(x_1)[\bar{S}_2(x - x_1) - \psi\bar{S}_4(x - x_1)]; \qquad x_1 \le x < x_2 \tag{23}$$

We can have *n* cracks with the following shape function

$$y_i(x) = y_1(x) + \sum_{j-1}^{i-1} C_j y_i^{''}(x_i)[\bar{S}_2(x - x_i) - \psi \bar{S}_4(x - x_i)]; \qquad x_{i-1} \le x < x_i \tag{24}$$

$$y_{i+1}(x) = y_1(x) + \sum_{j=1}^{n} C_j y_i^{''}(x_i)[\bar{S}_2(x - x_i) - \psi \bar{S}_4(x - x_i)]; \qquad x_n \le x < x_{n+1} \tag{25}$$

3. Result and discussion

In this section, numerical results are given for analytical solutions presented in the previous section. The main aim is to study the influences of thermal effect, nonlocal parameter, crack location, crack severity and type of boundary condition on the natural frequencies of the nanotube. For a nanotube with simply supported end position, the following relations must be satisfied

$$y(0) = y^{''}(0) = y(1) = y^{''}(1) = 0. \tag{26}$$

For example, when we consider two similar cracks in the nanotube, the mode shape functions for all segments are

$$y_1(x) = y_1^{'}(0)\bar{S}_2(x) + y_1^{'''}(0)\bar{S}_4(x). \tag{27a}$$

$$y_2(x) = y_1(x) + C_1 y_1^{''}(x_1)[\bar{S}_2(x - x_1) - \psi \bar{S}_4(x - x_1)]. \tag{27b}$$

$$y_3(x) = y_2(x) + C_2 y_2^{''}(x_2)[\bar{S}_2(x - x_2) - \psi \bar{S}_4(x - x_2)]. \tag{27c}$$

When applying the boundary condition at $x = 1.$ have (Binici 2005)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} y_1^{'}(0) \\ y_1^{'''}(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{28}$$

Where coefficients a_{11} through a_{22} are given in Appendix A.1. For achieving the non-trivial solution, the determinant of the coefficient matrix [a] must be equal to zero. Responses of the determinant are dimensionless natural frequencies of the nanotube. Now the results obtained by using the method described in this paper will be evaluated with the results obtained by Phadikar and Pradhan (2009) to check the accuracy of the above equations. In order to perform a parametric study, we consider the following value of the dimensionless parameters:

$C_i = 0. E = 1. I = 1. L = 1.$ $\rho = 1$ (Roostai and Haghpanahi 2014) and $\nu = 0.3$ (Falvo *et al.* 1997).

All the parameters have been considered 1, it means these parameters are dimensionless and useful for all the materials and geometries. At temperature equal to or lower than room temperature, $\alpha = -1.6 \times 10^{-6}$, the values of temperature change considered are $T = 0K. T = 100K$ and $T = 200K$. At temperature higher than room temperature, $\alpha = 1.1 \times 10^{-6}$, the values of temperature change considered are $T = 400K. T = 600K$ (Yao and Han 2006).

Table 1 The comparison of natural frequencies of the simply supported nanotube without cracks for different methods

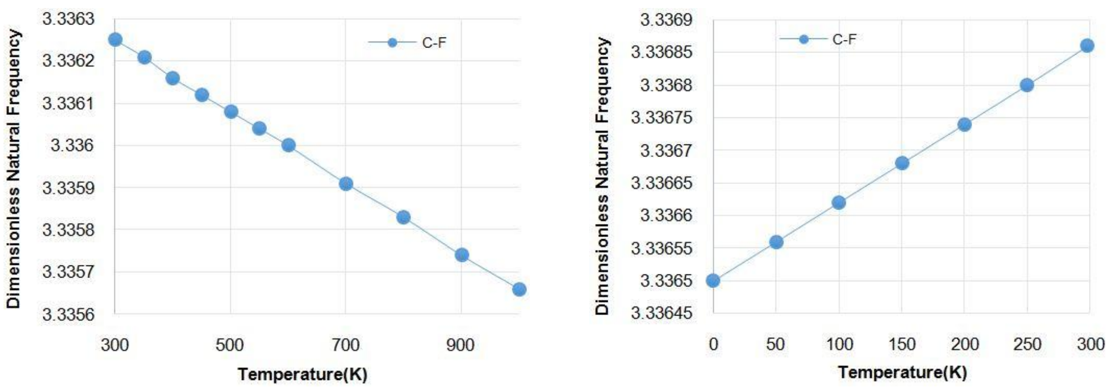
ε	Mode No.	Natural frequency (Pardha and Phadikar 2009)	Natural frequency (present)
0	1	9.8696	9.86960440
0	2	39.4784	39.47841760
0	3	88.8249	88.82643960
1	1	2.9936	2.99359383
1	2	6.2051	6.20508840
1	3	9.3720	9.37217008

In Table 1, value of dimensionless natural frequencies of the simply supported nanotube are given for intact nanotube and so without thermal effects, they are for the first three modes. Also, the data on this table were compared with Phadikar and Pradhan (2009). The table shows that the presented results are in suitable agreement with the previous study. Table 2 displays the values of the first three dimensionless natural frequencies of the simply supported nanotube to show the thermal effect on different nonlocal parameter ε and crack severity C_i, where the nanotube (L = 1 nm) has two similar cracks at $x_{1.2} = 0.3$ and 0.7. From Table 2, it can be noticed that an increase in crack severity decreases the natural frequency parameter of the cracked nanotube. Also, it is observable that an increase in the nonlocal parameter causes a decrease in the natural frequency as expected. Moreover, it can be observed that, when temperatures are equal or lower than room temperature the natural frequency decrease with increasing temperature.

Although, when the temperatures are higher than room temperature, natural frequency increases with increasing temperatures. This behavior is because of the difference of heat transfer coefficient for listed temperatures. That is why there is a fluctuation in temperature change curve, and it is established for all three natural frequencies. In the following, there are same tables for other boundary conditions. Using classical boundary conditions, like clamped–clamped, clamped–simply supported and clamped–free, the effect of them on natural frequency shows in Table 3-4-5, respectively. Coefficients a_{11} through a_{22} for these boundary conditions have been presented in Appendices A.2-A.4. It can be seen in these table that variations in the coefficients of clamped–clamped and clamped–simply supported is similar to changing in simply supported. Also, the natural frequency in the mentioned boundary conditions has same behavior to the simply supported. It means natural frequency decrease with increasing temperature below room temperature and increase with increasing the temperature at temperature above room temperature. But it can be seen that we have some different variations for the case of clamped–free nanotube results. Figs. 3 and 4 shows the thermal effect on values of the first and second dimensionless natural frequencies of the clamped–free, where crack severity is $C_i = 0.075$. And, we consider this crack severity because of the small size of the nanobeam. It has opposite behavior on the first natural frequency, that's implies first natural frequencies are increasing whit increasing of the temperature, when the temperatures are equal or lower than room temperature. But when the temperatures are higher than room temperature, natural frequency increases with increasing temperature. For second and third natural frequencies this boundary condition has same variation to other. This difference is due to the free edge of a nanobeam that causes the difference of the behavior of the first frequencies in comparison with other boundary conditions. Challamel and

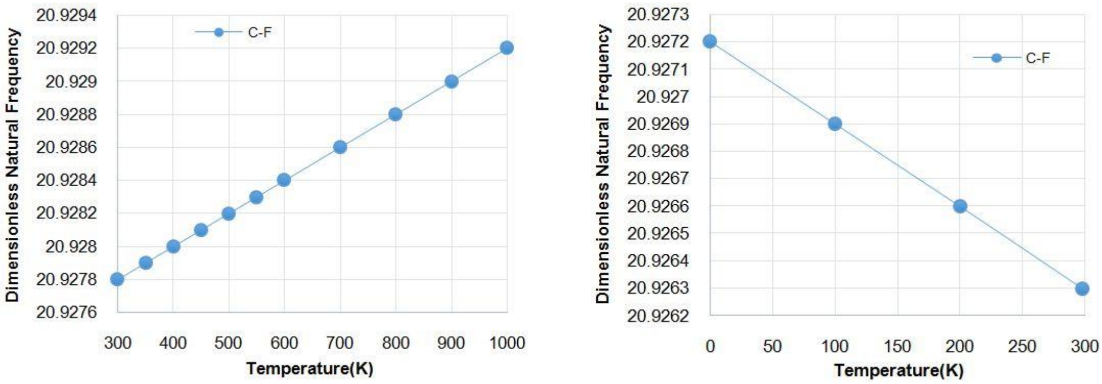
Table 2 Values of the thermal effect on first three dimensionless natural frequencies of the simply supported cracked nanotube for different crack severity and different nonlocal parameters

ε	T	$C_i = 0$			$C_i = 0.0086$			$C_i = 0.0325$			$C_i = 0.075$		
		ω			ω			ω			ω		
		1	2	3	1	2	3	1	2	3	1	2	3
0	0	9.8696	39.4784	88.8249	9.76033	38.8776	88.6833	9.47461	37.3365	88.3131	9.02284	34.9846	87.7404
	100	9.8694	39.4782	88.8262	9.76013	39.8774	88.6831	9.47442	37.3363	88.3129	9.02266	34.9845	87.7400
	200	9.8692	39.4780	88.8260	9.75993	39.8772	88.6829	9.47423	37.3362	88.3127	9.02247	34.9843	87.7402
	400	9.8711	39.4799	88.8279	9.76181	39.8790	88.6848	9.47605	37.3380	88.3146	9.02420	34.9860	87.7419
	600	9.87185	39.4807	88.8287	9.76255	39.8798	88.6856	9.476677	37.3387	88.3153	9.02489	34.9866	87.7426
0.5	0	5.30027	11.9744	18.4390	5.24146	11.7906	18.4086	5.08669	11.3066	18.3236	4.83945	10.5416	18.1721
	100	5.29989	11.9737	18.4380	5.2411	11.7900	18.4076	5.08635	11.3061	18.3227	4.83914	10.5411	18.1707
	200	5.29952	11.9731	18.4370	5.24074	11.7893	18.4067	5.08601	11.3055	18.3217	4.83883	10.5407	18.1698
	400	5.30305	11.9793	18.4462	5.24419	11.7953	18.4158	5.08925	11.3108	18.3307	4.84177	10.5451	18.1785
	600	5.30444	11.9818	18.4498	5.24555	11.7977	18.4193	5.09053	11.313	18.3342	4.84293	10.5468	18.182
1	0	2.9936	6.2051	9.3720	2.96036	6.10979	9.35673	2.87273	5.85828	9.31345	2.73234	5.45963	9.23584
	100	2.99293	6.20382	9.37027	2.95972	6.10857	9.35485	2.87214	5.8571	9.31159	2.73182	5.45876	9.23402
	200	2.99227	6.20254	9.36838	2.95908	6.10735	9.35296	2.87155	5.85613	9.30973	2.7313	5.4579	9.23221
	400	2.99852	6.21459	9.38633	2.96514	6.11886	9.37082	2.87715	5.86629	9.32734	2.7362	5.46609	9.24938
	600	3.00098	6.21934	9.39340	2.96753	6.12341	9.37786	2.87935	5.87029	9.33428	2.73813	5.46931	9.25614
1.5	0	2.04877	4.16541	6.26753	2.02603	4.10143	6.2572	1.96602	3.93249	6.22825	1.8698	3.66458	6.17632
	100	2.04781	4.16351	6.26469	2.02509	4.09962	6.25438	1.96516	3.9309	6.22547	1.86902	3.6633	6.17361
	200	2.04685	4.16162	6.26185	2.02416	4.0978	6.25156	1.96430	3.92931	6.22268	1.86832	3.66202	6.1709
	400	2.05596	4.17955	6.28868	2.03299	4.11493	6.27825	1.97241	3.94437	6.2490	1.87534	3.6741	6.19655
	600	2.04877	4.16541	6.26753	2.03646	4.12167	6.28875	1.97559	3.95029	6.25935	1.87809	3.67885	6.20633



(a) (b)

Fig. 3 Thermal effect on first dimensionless natural frequency of clamped-free



(a) (b)

Fig. 4 Thermal effect on second dimensionless natural frequency of clamped-free

Table 3 Values of the thermal effect on first three dimensionless natural frequencies of the clamped-clamped cracked nanotube for different crack severity and different nonlocal parameter

ε	T	$C_i = 0$			$C_i = 0.0086$			$C_i = 0.0325$			$C_i = 0.075$		
		ω			ω			ω			ω		
		1	2	3	1	2	3	1	2	3	1	2	3
	0	22.3733	61.6728	120.903	22.3175	60.7699	120.263	22.1735	58.5131	118.632	21.9518	55.2254	116.182
	100	22.3730	61.6724	120.903	22.3172	60.7695	120.263	22.1732	58.5127	118.631	21.9515	55.2251	116.182
0	200	22.3727	61.6720	120.903	22.3169	60.7691	120.262	22.1730	58.5123	118.631	21.9513	55.2247	116.181
	400	22.3741	61.6739	120.905	22.3183	60.7709	120.264	22.1743	58.5141	118.633	21.9525	55.2263	116.184
	600	22.3745	61.6745	120.905	22.4187	60.7715	120.265	22.1745	58.5146	118.633	21.9529	55.2268	116.184
	0	10.9914	17.273	24.3324	10.9705	16.9854	24.0854	10.9145	16.2469	23.4133	10.8218	15.1382	22.3244
	100	10.9897	17.2705	24.3290	10.9689	16.9831	24.0822	10.9130	16.245	23.4105	10.8204	15.1367	22.3221
0.5	200	10.9881	17.268	24.3256	10.9673	16.9807	24.0790	10.9114	16.243	23.4077	10.8189	15.1352	22.3199
	400	10.9959	17.2705	24.3417	10.975	16.9918	24.0943	10.9188	16.2524	23.4211	10.8259	15.1424	22.3305
	600	10.9982	17.268	24.3464	10.772	16.995	24.0987	10.9210	16.2551	23.4249	10.8279	15.1444	22.336
	0	6.05656	8.89543	12.453	6.04615	8.89543	12.453	6.04615	8.74626	12.3173	6.01792	7.78563	11.3556
	100	6.05318	8.89054	12.4462	6.04282	8.89054	12.4462	6.04282	8.74166	12.3190	6.01473	7.78276	11.3515
1	200	6.0498	8.88564	12.4394	6.03950	8.88564	12.4394	6.03950	8.73707	12.3044	6.01154	7.77989	11.3473
	400	6.06583	8.90889	12.4717	6.05529	8.90889	12.4717	6.05529	8.75888	12.3350	6.02668	7.7935	11.367
	600	6.07047	8.91561	12.4811	6.05985	8.91561	12.4811	6.05985	8.76518	12.3438	6.03105	7.79743	11.3727
	0	4.1191	5.96392	8.34304	4.1128	5.86376	8.25062	4.09405	5.60608	7.99968	4.0622	5.21847	7.59685
	100	4.1146	5.95658	8.3328	4.10778	5.85688	8.24098	4.08924	5.60032	7.99155	4.05776	5.20993	7.58459
1.5	200	4.10949	5.94923	8.32255	4.10275	5.84999	8.23133	4.08442	5.59455	7.98341	4.0533	5.20993	7.58459
	400	4.13369	5.9845	8.37114	4.12658	5.88264	8.27707	4.10724	5.62187	8.02195	4.07438	5.23015	7.61463
	600	4.14066	5.9941	8.38515	4.13345	5.89205	8.29025	4.11381	5.62974	8.03304	4.08045	5.23596	7.62199

Wang (2008) study some simplified non-local elastic cantilever beam models, for the bending analyses of small scale rods. And they focused on clamped – free behavior of the nanobeam and show these paradoxes.

It has opposite behavior on the first natural frequency, that's implies first natural frequencies are increasing whit increasing of the temperature, in temperatures are equal or lower than room

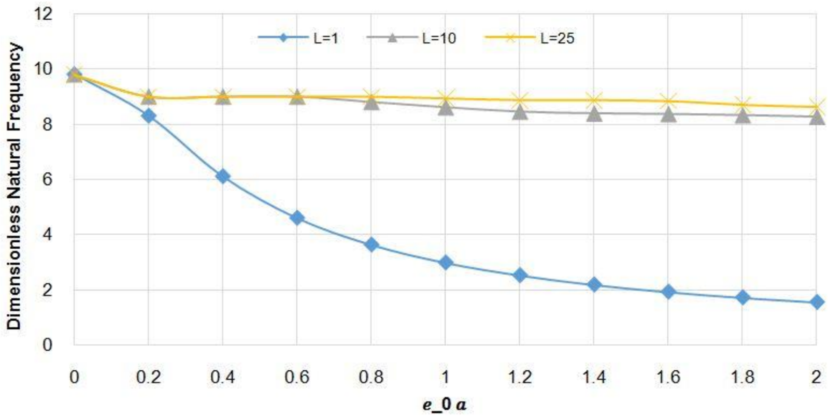


Fig. 5 The dimensionless natural frequency vs. temperature for various nanobeam lengths, where the nanobeam has two similar cracks

Table 4 Values of the thermal effect on first three dimensionless natural frequencies of the clamped-simply supported cracked nanotube for different crack severity and different nonlocal parameter

ε	T	$C_i = 0$			$C_i = 0.0086$			$C_i = 0.0325$			$C_i = 0.075$		
		ω			ω			ω			ω		
		1	2	3	1	2	3	1	2	3	1	2	3
	0	15.4182	49.9649	104.248	15.3126	47.7009	102.626	15.0336	47.7009	102.626	14.583	45.2639	100.955
	100	15.4178	49.9644	104.247	15.3122	47.7004	102.625	15.0332	47.7004	102.625	14.5826	45.2635	100.954
0	200	15.4147	49.9639	104.247	15.3118	47.7	102.625	15.0328	47.7	102.625	14.5822	45.2613	100.954
	400	15.4193	49.9661	104.249	15.3137	47.7021	102.627	15.0347	47.7021	102.627	14.584	45.526	100.956
	600	15.1499	49.9668	104.250	15.3142	47.7026	102.628	15.0352	47.7026	102.628	14.51845	45.2655	100.957
	0	7.7837	14.6881	21.2563	7.72158	14.5307	20.6796	7.55369	14.111	20.6796	7.27188	13.2656	20.0656
	100	7.78243	14.6809	21.2533	7.72035	14.5286	20.6769	7.55253	14.1091	20.6769	7.954	13.4396	20.0632
0.5	200	7.78116	14.6838	21.2503	7.71911	14.5266	20.6742	7.55136	14.1073	20.6742	7.26979	13.438	20.0608
	400	7.78718	14.6939	21.2645	7.72499	14.5363	20.6869	7.55689	14.1161	20.6869	7.27476	13.4454	20.0722
	600	7.78892	14.6969	21.2686	7.72669	14.5392	20.6906	7.55849	14.1187	20.6906	7.27619	13.4476	20.0755
	0	4.31824	7.62114	10.830	4.28216	7.54426	10.7428	4.48411	7.33849	10.5216	4.01833	7.00842	10.1941
	100	4.31578	7.61693	10.824	4.27976	7.5402	10.7371	4.1819	7.33482	10.5163	4.0164	7.00536	10.1895
1	200	4.31331	7.61271	10.8181	4.27736	7.53613	10.7313	4.17968	7.33114	10.511	4.01446	7.00231	10.1848
	400	4.3250	7.63272	10.8463	4.28874	7.55543	10.7586	4.1902	7.34857	10.536	4.02363	7.0168	10.2069
	600	4.32839	7.6385	10.8544	4.29202	7.56101	10.7664	4.9324	7.3536	10.5433	4.02627	7.02098	10.2133

Table 4 Continued

ε	T	$C_i = 0$			$C_i = 0.0086$			$C_i = 0.0325$			$C_i = 0.075$		
		ω			ω			ω			ω		
		1	2	3	1	2	3	1	2	3	1	2	3
	0	2.94206	5.11872	7.24708	2.91718	5.06778	7.18814	2.84953	4.93175	7.03851	2.73495	4.71317	6.81716
	100	2.93828	5.11241	7.23818	2.91361	5.0618	7.17954	2.84624	4.92626	7.033064	2.7321	4.7086	6.81023
1.5	200	2.93469	5.10609	7.22926	2.91003	5.0557	7.17093	2.84294	4.92075	7.02275	2.72925	4.70402	6.80328
	400	2.95215	5.13604	7.27151	2.92698	5.0846	7.21173	2.85855	4.94638	7.06009	2.74247	4.72568	6.83617
	600	2.95719	5.14468	7.28369	2.93187	5.09293	7.22349	2.86305	4.95434	7.07085	2.74662	4.7319	6.84564

Table 5 Values of the thermal effect on first three dimensionless natural frequencies of the clamped-clamped nanotube, which including three similar cracks for different crack severity and different nonlocal parameter

ε	T	$C_i = 0$			$C_i = 0.0086$			$C_i = 0.0325$			$C_i = 0.075$		
		ω			ω			ω			ω		
		1	2	3	1	2	3	1	2	3	1	2	3
	0	22.3733	61.6728	120.903	22.2256	61.06	-	21.8342	57.7955	-	21.2052	52.1948	-
	100	22.3730	61.6724	120.903	22.2253	61.0596	-	21.8339	57.7953	-	21.2050	52.1947	-
0	200	22.3727	61.6720	120.903	22.225	61.0592	-	21.8337	57.7950	-	21.2047	52.1947	-
	400	22.3741	61.6739	120.905	22.2264	61.0611	-	21.8350	57.7962	-	21.2059	52.1950	-
	600	22.3745	61.6745	120.905	22.2268	61.0616	-	21.8354	57.7965	-	21.2063	52.1951	-
	0	10.9914	17.273	24.3324	10.8936	17.2095	23.8083	10.6502	17.0148	22.4420	10.3128	16.6327	20.8942
	100	10.9897	17.2705	24.3290	10.892	17.2071	23.8051	10.6487	17.0125	22.4349	10.3115	16.6307	20.8920
0.5	200	10.9881	17.268	24.3256	10.8904	17.2047	23.802	10.6472	17.0103	22.4369	10.3102	16.6287	20.8898
	400	10.9959	17.2797	24.3417	10.898	17.2161	23.8169	10.6543	17.0210	22.4490	10.3165	16.638	20.9003
	600	10.9982	17.2831	24.3464	10.9002	17.2195	23.8212	10.6563	17.0241	22.4525	10.3184	16.6407	20.9033
	0	6.05656	8.89543	12.453	5.99849	8.86409	12.1942	5.8565	8.76709	11.5096	5.66933	8.57135	10.7247
	100	6.05318	8.89054	12.4462	5.99523	8.85929	12.1879	5.8535	8.76261	11.5045	5.66667	8.56754	10.7203
1	200	6.0498	8.88564	12.4394	5.99196	8.85449	12.1816	5.85051	8.75812	11.4994	5.66401	8.56373	10.7160
	400	6.06583	8.90889	12.4717	6.00746	8.87727	12.2116	5.86471	8.77938	11.5236	5.67662	8.5818	10.7366
	600	6.07047	8.91561	12.4811	6.01194	8.88386	12.2202	5.86881	8.78553	11.5306	5.68027	8.58701	10.7426
	0	4.1191	5.96392	8.34304	4.07949	5.94308	8.17131	3.98185	5.87847	7.71542	3.85415	5.74717	7.19087
	100	4.1146	5.95658	8.3328	4.07457	5.93589	8.16183	3.97708	5.87847	7.70778	3.85017	5.74149	7.18437
1.5	200	4.10949	5.94923	8.32255	4.06964	5.92869	8.15234	3.97258	5.86505	7.70013	3.84618	5.7358	7.17787
	400	4.13369	5.9845	8.37114	4.09299	5.96281	8.19732	3.99389	5.89686	7.73635	3.86505	5.76271	7.2087
	600	4.14066	5.9941	8.38515	4.09972	5.97265	8.21028	4.00003	5.90603	7.74677	3.87048	5.77044	7.21759

temperature. But it is decreased in the higher room temperature when its increase. For second and third natural frequencies this boundary condition has same variation to other. Table 5 shows thermal effect on values of the first three dimensionless natural frequencies of the clamped-

clamped nanotube, which have three similar cracks for different values of nonlocal parameter and crack severity. For $C_i = 0$ this table is same to Table 3, because it implies the nanotube has not any crack. But for the other crack severity, it can be seen that natural frequency decrease with increasing temperature, below room temperature. Also, it increased with increasing temperature, above room temperature. Fig. 5 shows the dimensionless natural frequency of the simply supported nanobeam ($h = 0.1$ nm, $\rho = 2700h^2$, $I = h^4/12$ and $E = 70$ Gpa) vs. e_0a for various nanobeam lengths, where the nanobeam has two similar cracks ($C_i = 0.0086$) at $x_{1.2} =$ 0.2 and 0.8, when we have not thermal effect. It can be noticed that as the length of the nanobeam increases the effect of e_0a on the dimensionless natural frequency decreases.

Fig. 6 displays the dimensionless natural frequency of the simply supported nanotube vs. temperature for various locations of cracks (both cracks are similar ($C_i = 0.075$)). The nonlocal parameter and length for this figure are $e_0a = 0$ and $L = 1$, respectively. Also this figure shows the natural frequency decrease as the distance between cracks decrease. And, this figures show when the cracks are near to each other the dimensionless natural frequencies are lesser than the cracks are far away.

Figs. 7 and 8 present the thermal effect on the first and second dimensionless natural frequency of nanotubes of different boundary conditions, respectively. Both of these tables have similar crack

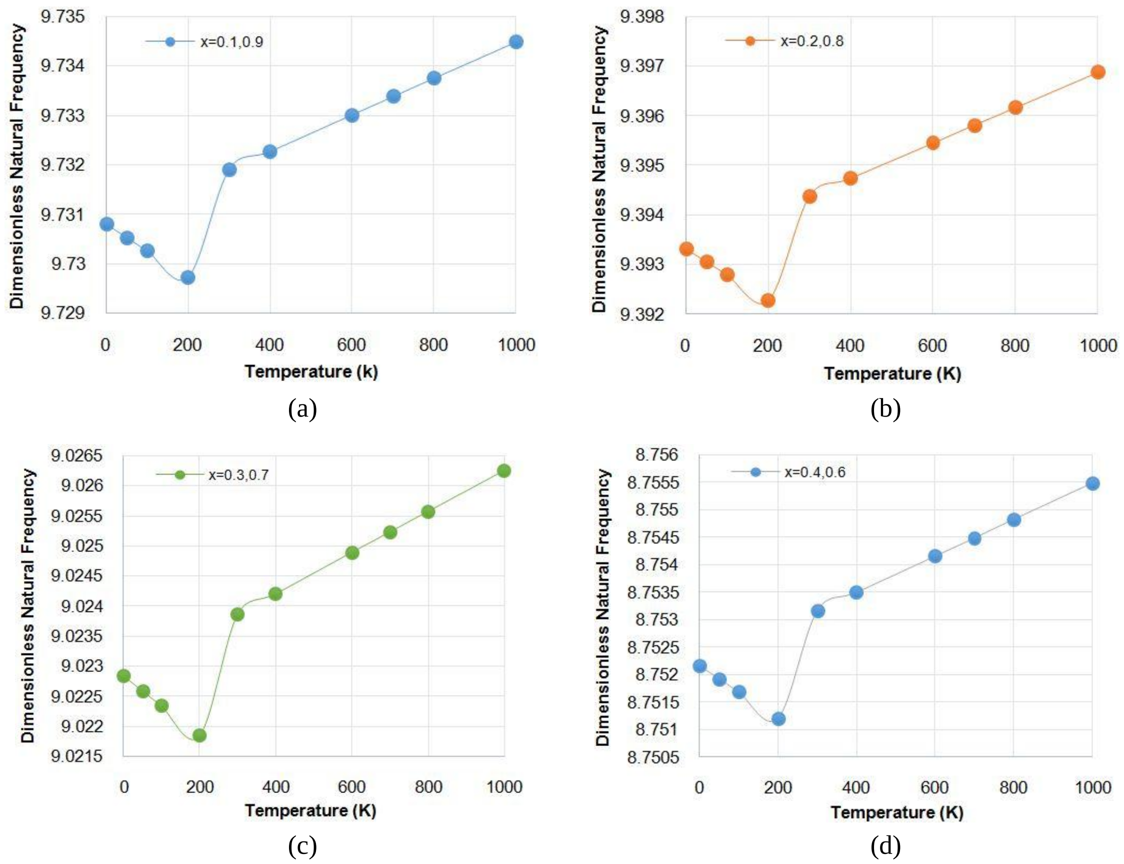


Fig. 6 The dimensionless natural frequency of the simply supported nanotube vs. temperature for various locations of cracks, where the nanotube has two similar cracks

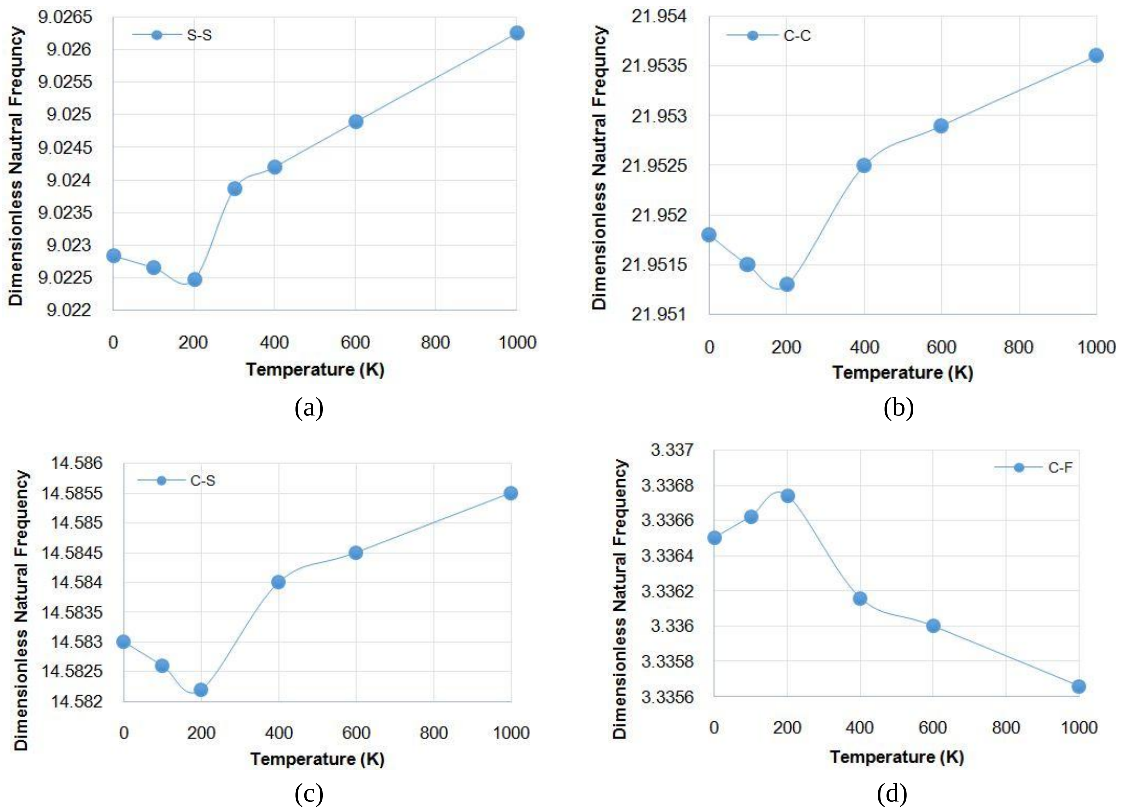


Fig. 7 Thermal effect on the first dimensionless natural frequency of nanotubes of different boundary conditions. ($C_i = 0.075$) (one by one)

with crack severity $C_i = 0.075$. It vivid that clamped-clamped have the highest values of natural frequency and clamped- free have the lowest. Also temperature have the most influence on simply supported, that mean non dimensional natural frequencies of this beam have more alteration when

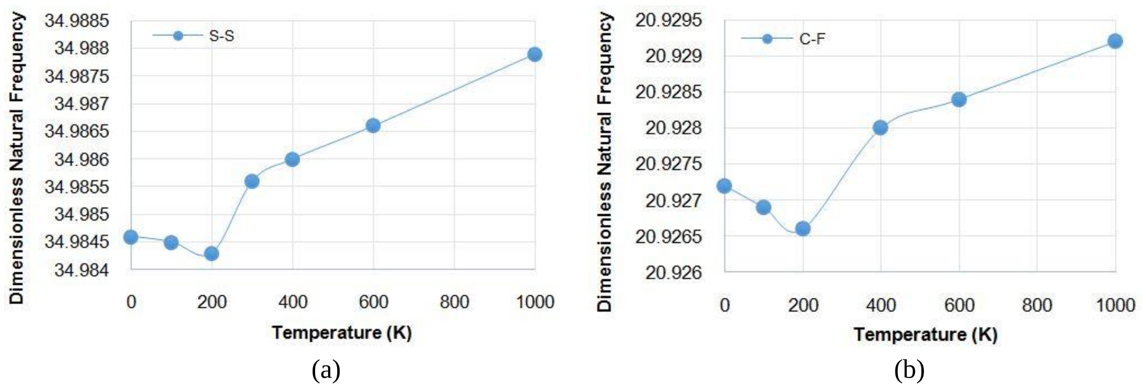
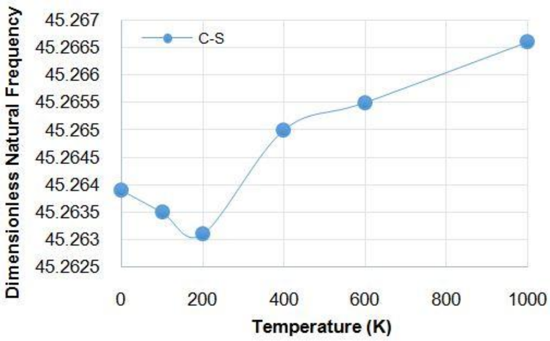
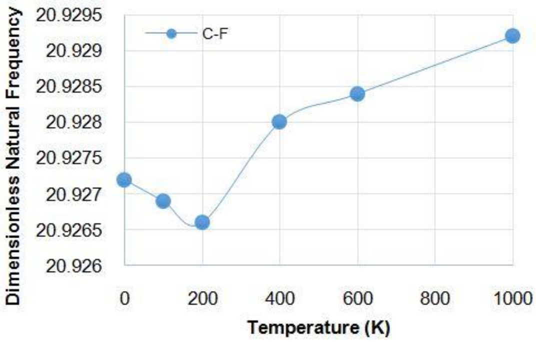


Fig. 8 Thermal effect on the second dimensionless natural frequency of nanotubes of different boundary conditions. ($C_i = 0.075$)



(c)



(d)

Fig. 8 Continued

the boundary condition is simply supported. And they have light change when the beam is on clamp – clamp boundary condition. About clamp – free we deal with singular effect that in the second natural frequency, this dimensionless natural frequency rising with temperature rise and in the first dimensionless natural frequency it's reduce with temperature rising.

4. Conclusions

The main objective of this paper is to examine the thermal effects on the free vibration behavior of nanotube that have multiple cracks for different boundary conditions. The temperature variations, different length of the nanobeam, various positions of the crack, different crack severities and different boundary conditions can change the values of natural frequencies. It can be concluded the crack has reduction effect on natural frequency. It can be noticed, for most of the boundary conditions, when temperatures are equal or lower than room temperature the natural frequency decrease with increasing, when the temperatures are higher than room temperature, natural frequency increase with increasing temperature. As presented in this paper, the effects of nonlocal parameter on the dimensionless natural frequency are apparent when the length of the nanobeam is in nanoscale and in longer length the natural frequency converge to the local frequency. Moreover, it can be seen the crack position have its effect on natural frequency, it is obvious that when the cracks poisons are near to each other natural frequencies decrease. All of different crack position curves have same behavior. Raising the natural frequency with crack severity raise was displayed. In different boundary conditions, clamped-clamped has larger natural frequencies. At the end influence of number of cracks on clamed-clamped was showed, it can be noticed that when the number of cracks raise the natural frequencies decrease.

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Appendix

A.1 Simply supported beam

$$a_{11} = \bar{S}_2(1) + C_1\bar{S}_2''(x_1)T(1-x_1) + C_2T(1-x_2)[\bar{S}_2''(x_2) + C_1\bar{S}_2''(x_1)T''(x_2-x_1)],$$
$$a_{12} = \bar{S}_4(1) + C_1\bar{S}_4''(x_1)T(1-x_1) + C_2T(1-x_2)[\bar{S}_4''(x_2) + C_1\bar{S}_4''(x_1)T''(x_2-x_1)],$$
$$a_{21} = \bar{S}_2''(1) + C_1\bar{S}_2''(x_1)T(1-x_1) + C_2T''(1-x_2)[\bar{S}_2''(x_2) + C_1\bar{S}_2''(x_1)T''(x_2-x_1)],$$
$$a_{22} = \bar{S}_4''(1) + C_1\bar{S}_2''(x_1)T(1-x_1) + C_2T''(1-x_2)[\bar{S}_4''(x_2) + C_1\bar{S}_4''(x_1)T''(x_2-x_1)].$$

Where

$$T(x) = \bar{S}_2(x) - \psi\bar{S}_4(x)$$

A.2 Clamped–clamped beam

$$a_{11} = \bar{S}_3(1) + C_1\bar{S}_3''(x_1)T(1-x_1) + C_2T(1-x_2)[\bar{S}_3''(x_2) + C_1\bar{S}_3''(x_1)T''(x_2-x_1)],$$
$$a_{12} = \bar{S}_4(1) + C_1\bar{S}_4''(x_1)T(1-x_1) + C_2T(1-x_2)[\bar{S}_4''(x_2) + C_1\bar{S}_4''(x_1)T''(x_2-x_1)],$$
$$a_{21} = \bar{S}_3'(1) + C_1\bar{S}_3''(x_1)T(1-x_1) + C_2T'(1-x_2)[\bar{S}_3''(x_2) + C_1\bar{S}_3''(x_1)T''(x_2-x_1)],$$
$$a_{22} = \bar{S}_4'(1) + C_1\bar{S}_2''(x_1)T(1-x_1) + C_2T'(1-x_2)[\bar{S}_4''(x_2) + C_1\bar{S}_4''(x_1)T''(x_2-x_1)].$$

A.3 Clamped–simply supported beam

$$a_{11} = \bar{S}_3(1) + C_1\bar{S}_3''(x_1)T(1-x_1) + C_2T(1-x_2)[\bar{S}_3''(x_2) + C_1\bar{S}_3''(x_1)T''(x_2-x_1)],$$
$$a_{12} = \bar{S}_4(1) + C_1\bar{S}_4''(x_1)T(1-x_1) + C_2T(1-x_2)[\bar{S}_4''(x_2) + C_1\bar{S}_4''(x_1)T''(x_2-x_1)],$$
$$a_{21} = \bar{S}_3''(1) + C_1\bar{S}_3''(x_1)T''(1-x_1) + C_2T''(1-x_2)[\bar{S}_3''(x_2) + C_1\bar{S}_3''(x_1)T''(x_2-x_1)],$$
$$a_{22} = \bar{S}_4''(1) + C_1\bar{S}_2''(x_1)T''(1-x_1) + C_2T''(1-x_2)[\bar{S}_4''(x_2) + C_1\bar{S}_4''(x_1)T''(x_2-x_1)].$$

A.4 Clamped–free beam

$$a_{11} = \bar{S}_3''(1) + C_1\bar{S}_3''(x_1)T''(1-x_1) + C_2T''(1-x_2)[\bar{S}_3''(x_2) + C_1\bar{S}_3''(x_1)T''(x_2-x_1)],$$
$$a_{12} = \bar{S}_4''(1) + C_1\bar{S}_4''(x_1)T''(1-x_1) + C_2T''(1-x_2)[\bar{S}_4''(x_2) + C_1\bar{S}_4''(x_1)T''(x_2-x_1)],$$
$$a_{21} = \bar{S}_3'''(1) + C_1\bar{S}_3''(x_1)T'''(1-x_1) + C_2T'''(1-x_2)[\bar{S}_3''(x_2) + C_1\bar{S}_3''(x_1)T''(x_2-x_1)],$$
$$a_{22} = \bar{S}_4'''(1) + C_1\bar{S}_2''(x_1)T'''(1-x_1) + C_2T'''(1-x_2)[\bar{S}_4''(x_2) + C_1\bar{S}_4''(x_1)T''(x_2-x_1)].$$

A.5 Clamped–clamped beam (*three cracks*)

$$a_{11} = \bar{S}_3(1) + C_1\bar{S}_3''(x_1)T(1-x_1) + C_2T(1-x_2)[\bar{S}_3''(x_2) + C_1\bar{S}_3''(x_1)T''(x_2-x_1)]$$
$$+C_3T(1-x_3)[\bar{S}_3''(x_3) + C_1\bar{S}_3''(x_1)T(x_3-x_1) + C_2\left(\bar{S}_3''(x_2)\right) + C_1\bar{S}_3''(x_1)T(x_2-x_1)],$$

$$a_{12} = \bar{S}_4(1) + C_1\bar{S}_4''(x_1)T(1-x_1) + C_2T(1-x_2)[\bar{S}_4''(x_2) + C_1\bar{S}_4''(x_1)T''(x_2-x_1)$$
$$+C_3T(1-x_3)[\bar{S}_4''(x_3) + C_1\bar{S}_4''(x_1)T(x_3-x_1) + C_2\left(\bar{S}_4''(x_2)\right) + C_1\bar{S}_4''(x_1)T(x_2-x_1)],$$

$$a_{21} = \bar{S}_3'(1) + C_1\bar{S}_3''(x_1)T(1-x_1) + C_2T'(1-x_2)[\bar{S}_3''(x_2) + C_1\bar{S}_3''(x_1)T''(x_2-x_1)$$
$$+C_3T'(1-x_3)[\bar{S}_3''(x_3) + C_1\bar{S}_3''(x_1)T(x_3-x_1) + C_2\left(\bar{S}_3''(x_2)\right) + C_1\bar{S}_3''(x_1)T(x_2-x_1)],$$

$$a_{22} = \bar{S}_4'(1) + C_1\bar{S}_2''(x_1)T(1-x_1) + C_2T'(1-x_2)[\bar{S}_4''(x_2) + C_1\bar{S}_4''(x_1)T''(x_2-x_1)$$
$$+C_3T'(1-x_3)[\bar{S}_4''(x_3) + C_1\bar{S}_4''(x_1)T(x_3-x_1) + C_2\left(\bar{S}_4''(x_2)\right) + C_1\bar{S}_4''(x_1)T(x_2-x_1)].$$